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THE 82529
ELEMENTS
OF
EUCLID

Explain'd,
in a *new*, but most *easie* method,

Together with
The Use of every Proposition through all
parts of the Mathematicks.

Written in *French*, by that Excellent Mathe-
matician,

F. CLAUD. FRANCIS MILLIET de CHALES,
of the *Society of JESUS*.

And now carefully done into *English*, and purg'd from a mul-
titude of Errors, which had escap'd in the Original.

OXFORD,
Printed by L. Lichfield, Printer to the University,
for Anthony Stephens, Bookseller near the
Theater in Oxford. 1685.



*Thomas Foley of Great
Witley Court in the County
of Worcester Esq^r.*



THE
P R E F A C E.

HAVING for a long time observ'd, that most of those, that take in hand the Elements of Euclid, are apt to dislike them, because they cannot presently discern, to what end those seemingly inconsiderable, and yet difficult Propositions, can conduce: I thought I should do an acceptable piece of service, in not only rendring them as easie as possible, but also adding to each Proposition a brief account of some Use, that is made of them in the other parts of the Mathematicks. In prosecuting which design, I have been oblig'd to change some Demonstrations, that seem'd too intricate and perplex'd, and above the ordinary capacity of Beginners, and to substitute others more intelligible in their stead. For the same reason, I have demonstrated the fifth Book after a method, much more clear, than that by Equimultiples, formerly used. I would nat be thought to have set down all the Uses, that may be

The Preface.

*made of these Propositions: to have donee that, would have oblig'd me to have compris'd the whole Mathematicks in this one Book; which would have render'd it both too large, and too difficult. But I have contented my self with the choice of such, as may serve to point out some of the Advantages they afford us, and are also in themselves most clear, and most easie to be apprehended. I have distinguish'd them by * Inverted Commas, that the Reader may know them; not desiring hee should dwell too long upon them, or labour too understand them perfectly at first, since they depend on the Principles of the other Parts.*

This therefore being the design of this small Treatise, I voluntary offer it to the publick, in an Age, whose Genius seems more addicted to the Mathematicks, than any that has preceded it.

* Instead of the Authors Italian character.

Eight Books of the Elements
of **EUCLID**, together
with the Use of the Propo-
sitions.

THE FIRST BOOK.

‘ **T**HE design of *EUCLID* in this Book
‘ is to lay down the First Principles of
‘ *Geometry*; and to do it methodically,
‘ he begins with *Definitions*, and the expli-
‘ cation of the most ordinary Terms. To these
‘ he adds some *Postulata*; and then propo-
‘ sing those known *Maxims*, in which natu-
‘ ral reason does instruct us, he pretends, not to
‘ advance a step farther without a *Demonstration*,
‘ but to convince every man, even the most ob-
‘ stinate, that will grant nothing, but what is ex-
‘ torted from him. In the first Proposition, he
‘ treats of *Lines*, and the different *Angles*, which
‘ are

are form'd by their concurrence ; and having occasion to compare divers *Triangles* together, in order to demonstrate the Properties of *Angles*, he makes that the business of the Eight first Propositions. Then follow some Practical Instructions, how to divide an *Angle* and a *Line* into two equal parts, and to draw a *Perpendicular*. Next he shows the properties of a *Triangle*, together with those of *Parallel Lines* ; and having thus finish'd the Explication of this first figure, he passes on to *Parallelograms*, teaching the manner of reducing any *Polygone*, or multangular figure into one more regular. Lastly, he finishes the first Book with that famous Proposition of *Pythagoras*, That in every rectangular *Triangle* the Square of the * *Base* is equal to the Squares of both the other *Sides*.

* He calls that the *Base*, which is commonly call'd the *Hypotenuse*, i.e. the Line that is opposite to the right Angle.

DEFINITIONS.

1. **A** Point is that which hath no parts.

This Definition must be understood in this sense : That quantity, which we conceive without distinguishing its parts, or so much as considering whether or no it has any, is a *Mathematical* point ; which is therefore very dif-

The First Book.

3

different from those of *Zeno*, which were sup-
pos'd to be absolutely indivisible, and there-
fore such, that we may reasonably doubt whe-
ther they are possible; but the former we can-
not doubt of, if we conceive them aright.

2. A Line is length without breadth.

The sense of this definition is the same
with the former, That quantity, which we
conceive as length, without reflecting on its
breadth or thickness, is that, which we under-
stand by a Line; though it be impossible to
draw a real Line, which will not be of a cer-
tain breadth. 'Tis commonly said, that a Line
is produc'd by the motion of a Point; which
ought to be carefully observ'd; for motion
may on that manner produce any quantity
whatsoever: but here, we must imagine a
Point to be only so mov'd, as to leave one
trace in the space, through which it passes, and
then, that trace will be a line.

3. The two Extrems of a Line are Points.

4. A right Line is that, whose points are equal- ly plac'd between the two Extrems.

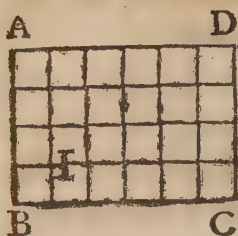
Or thus. A right Line is the shortest that
can be drawn from one point to another. Or
yet. The Extrems of a right Line may cast a
shadow upon the whole Line.

5. A Superficies or Surface is a quantity, to which is attributed length, and breadth, without the consideration of any thickness.

6. The

6. The Extrems of a Superficies are Lines.

7. A plane or right Superficies is that, whose lines are equally plac'd between its two Extrems; Or that, to which a right line may be every way apply'd.

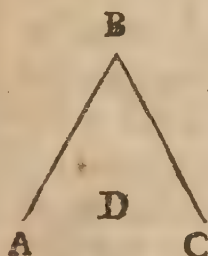


I have before observ'd, that
 ' motion may produce any quan-
 ' tity whatsoever: accordingly
 ' we say, when one line moves
 ' over another, it produces a Su-
 ' perficies, or a Plane; and that
 ' that motion has a kind of affinity with *Arith-*
 ' metical Multiplication. Suppose then the
 ' line AB to pass along the line BC, retaining
 ' still the same situation, without any inclination
 ' to one side or other: the point A will describe
 ' the line AD, the point B the line BC, and
 ' the intermediate points the lines parallel to
 ' those, which will make up the Superficies A
 ' B C D. I add further, that this motion answers
 ' to *Arithmetical* Multiplication; because did
 ' I know the number of points, that are contain'd
 ' in both those lines, AB, and DC; by multi-
 ' plying them together, I should find a product,
 ' which would give me the number of points,
 ' which constitute the whole superficies ABCD.
 ' As for example, if AB contain'd four points,
 ' and BC six, by saying four times six make
 ' twenty four, I find, that the whole superficies
 ' ABCD consists of twenty four points. Now
 by

‘by a *Mathematical* point, may be understood
 ‘any quantity whatsoever; e.g. a Foot, provi-
 ‘ded it be not subdivided into parts.

8. A plain Angle is the * distance or opening of
 two lines touching each other, so as not to compose
 only one line.

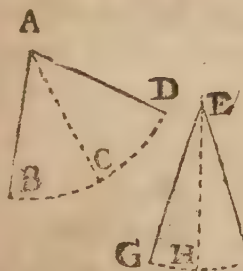
* Overture. Gall. πρὸς ἀλλήλας κλίσις. Eucl.]



‘As the distance D betwixt the
 ‘lines AB, and BC; which are not
 ‘parts of the same line.

9. A Rectilineal Angle is the di-
 stance betwixt two right lines.

‘Tis chiefly of this sort of An-
 ‘gles that I would be understood at present;
 ‘which I define by *distance or opening*, because
 ‘Experience teaches, that the greatest part of
 ‘Beginners deceive themselves in measuring
 ‘the greatness of an Angle by that of the lines,
 ‘within which it is contain’d.



‘The Angle that is more
 ‘open, is the greater; that
 ‘is, when the lines of one an-
 ‘gle lie more apart from each
 ‘other than those of another,
 ‘taking them at the same di-
 ‘stance from the points of concourse, the for-
 ‘mer is greater than the latter. Accordingly
 ‘the angle A is greater than the angle E; be-
 ‘cause

'cause taking the points D and B as remote
 'from the point A, as the points G and L are
 'from the point E; the points D and B lie far-
 'ther apart from each other, than the points G
 'and L: from whence I infer, that if the lines
 'EG and EL were produc'd farther, the angle
 'E would be always of the same largeness, and
 'always less than the angle A.

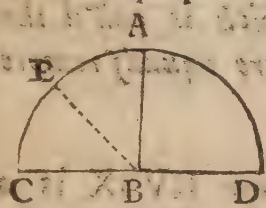
'We use three letters when we speak of an
 'Angle, of which the middlemost denotes the
 'point of concurrence; as the angle BAD is the
 'angle which by the lines BA and AD is form'd
 'at the point A: the angle BAC is that made
 'by the lines BA and AC: the angle CAD is
 'compris'd by the lines CA and AD.

'A Circle is the measure of an Angle. There-
 'fore to know the magnitude of the Angle
 'BAD, I place the foot of the Compass upon
 'the point A, and describe the circle BCD:
 'the angle is so much the greater, by how
 'many more parts of a circle the arch, that mea-
 'sures it, contains: and because a circle is usu-
 'ally divided into 360 parts, or degrees, there-
 'fore an angle is said to have twenty, thirty,
 'forty degrees, according as the arch, com-
 'pris'd betwixt the lines that form it, contains
 'so many. So the angle is the greater, which
 'contains more degrees, as the angle BAD is
 'greater than the angle GEL. 'The line CA
 'divides the angle BAD in the middle, because

thee

the arches BC and CD are equal: and the angle BAC is part of the angle BAD, because the arch BC is part of the arch BD.

10. When one line falling upon another makes two equal angles, they are both right angles; and the line perpendicular.



As for example: if the line AB, plac'd upon the line CD, make the angles ABC and ABD equal; that is, if, having describ'd a semicircle CAD from the center B, the arches AC and AD are equal: the angles ABC, and ABD are call'd right angles, and the line AB perpendicular. Therefore because the arch CAD is a semicircle, the arches CA and AD are each of them a quarter of a circle, that is the fourth part of three hundred and sixty degrees, that is ninety.

11. An Obtuse angle is that which is greater than a right one.

As the angle EBD is an obtuse or blunt angle, because its arch EAD contains more than a quarter of a circle.

12. An Acute angle is that which is less than a right one.

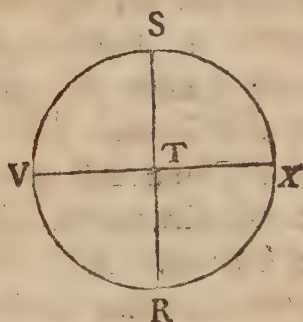
As the angle EBC is an acute, because the arch EC, which measures it, has less than ninety degrees.

13. A Term is the extremity or end of any quantity.

14. *A Figure is a quantity comprehended by one or more Terms.*

‘ That which is call’d a Figure ought to be
‘ limited and inclos’d on every side.

15. *A Circle is a plain figure, terminated by the encompassing of one line, which is call’d the Circumference; and is every where equally remote from the middle point.*



‘ The Figure RVSX is a
‘ Circle, because all the
‘ lines TR, TV, TS, TX,
‘ drawn from the point T
‘ to the line RVSX, are e-
‘ qual.

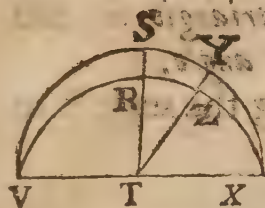
16. *The middle point is call’d the Center.*

17. *The Diameter of a Circle is any line passing through the Center, and terminated at the Circumference, dividing the Circle into two equal parts.*

‘ As the lines VTX, and RTS.

‘ But if any should doubt, whether the line
‘ VTX does indeed divide the circle into two
‘ equal parts, so that the part VSX be equal to
‘ the part VRX; it may on this manner be
‘ prov’d.

‘ Suppose the part VRX to be plac’d upon the
‘ other VSX: I say, they will not exceed one
‘ the

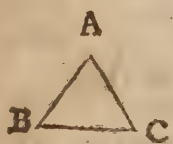


the other. For if one suppose
 VSX exceed the other VRX,
 the line TR will be less than
 TS; and in like manner TZ
 than TY, which is contrary to
 the definition of a Circle, which affirms all
 the lines drawn from the center to the circum-
 ference to be equal.

18. A Semicircle is a figure terminated by the
 Diameter, and half the Circumference.

19. Rectilineal figures are such as are termi-
 nated by right lines, having three, or four, or five,
 or as many sides as you please.

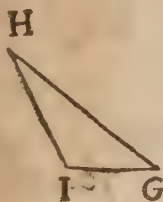
Euclid divides Triangles with respect ei-
 ther to their angles, or sides.



20. An Equilateral Triangle is
 that which has its three sides equal:
 ABC.

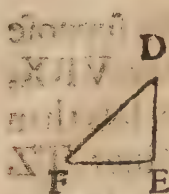


21. An Isosceles, or equicrural Tri-
 angle, is that which has two sides equal:
 As if the two sides AB, and AC be
 equal, the triangle ABC is an Iso-
 sceles.



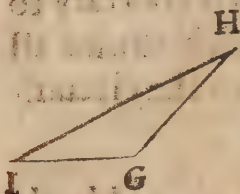
22. A Scalenum is a triangle hav-
 ing all the three sides unequal, as GHI.

23. A

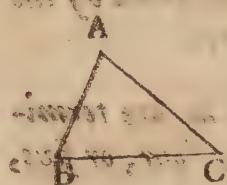


23. A Rectangle triangle is that which has one right angle.

As DEF, supposing the angle E to be a right one.



24. An Amblygone, or Obtuse angle triangle is that which has one angle obtuse. As IGH.



25. Oxygone, or Acute angle triangle is that whose angles are all acute. As ABC.



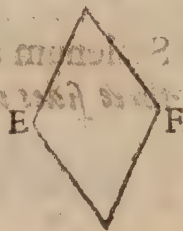
26. A Rectangle (properly so call'd) is a figure consisting of four sides, and having all its angles right.

27. A Square has all its sides equal, and its angles right, as AB.



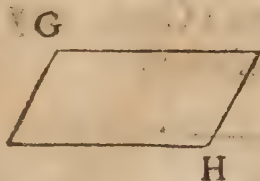
28. An Oblong Rectangle has its sides unequal, but its angles right: as CD.

29. A Rhombus, or Losange, has equal sides, but unequal angles: as EF.



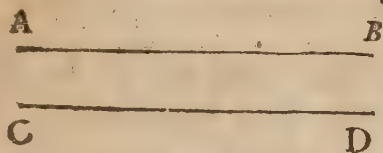
The First Book.

II

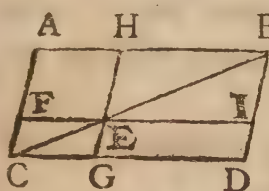


30. A Rhomboides, or oblong Lozenge, hath both its sides and angles unequal: as GH.

31. Other irregular figures of four sides are call'd Trapezia.



32. Parallel lines are such, as being in the same plane will never concur, keeping still an equal distance one from the other: as AB, CD.



33. A Parallelogram is a figure, whose two opposite sides are Parallels: as the Figure ABCD, whose sides AB, CD; and AC, BD, are parallels.

34. The Diameter of a Parallelogram is a right line drawn from one angle to another: as BC.

35. The Complements are the two small Parallelograms, through which the Diameter does not pass: as AFEH, and GDIE.

DEMANDS, or SUPPOSITIONS.

1. **T**IS suppos'd, that a right line may be drawn from any point whatsoever to another.

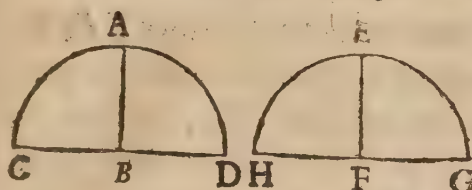
2. That a right line may be continu'd to what length you please.

3. That

3. That from a Center given a Circle may be describ'd at any distance whatsoever.

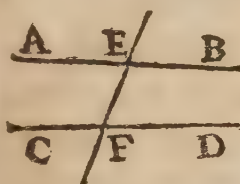
MAXIMES, or AXIOMES.

1. **T**Hose quantities that are equal to a third, are equal betwixt themselves.
2. If equal quantities be added to those that are equal, the products will also be equal.
3. If equal quantities be taken away from those that are equal, the remainders will be equal.
4. If you add equal parts to quantities unequal, they will remain unequal.
5. If from equal quantities you take away unequal parts, the remainders will be unequal.
6. Quantities that are double, triple, quadruple, &c. in respect of the same, are equal among themselves.
7. Those quantities are said to be equal, which being apply'd one to the other, neither exceeds.
8. Equal lines and angles being plac'd one upon another, do not surpass each other.
9. The whole is greater than its part.
10. All right angles are equal to one another.



Let the two
right angles pro-
pos'd be ABC,
EFH, I say they
are equal. For if

two equal circles CAD, HEG, be describ'd
from the centers B and F; the fourth parts of
those circles CA, HE, which are the measures
of the angles, ABC, EFH, will be equal:
therefore the angles ABC, EFH, having e-
qual measures, will be equal.

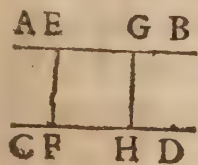


The eleventh Maxime of
Euclid is to this effect. If two
lines AB, CD, being cut by a
third EF, make the internal
angles, BEF, DFE, less than

two right angles; the lines AB, CD being pro-
duc'd, will at length concur towards the points
B and D.

Which, though it be true, is not clear
enough to be receiv'd for a Maxime: therefore
I have substituted another in its place.

11. If two lines be parallel, all the perpen-
diculars contain'd betwixt them will be equal.



As for example, if the lines AB,
CD, are parallels, the perpendi-
cular lines FE, HG, are equal.

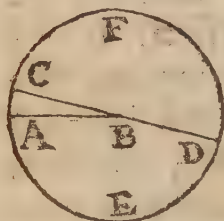
For if EF was greater than GH,
the lines AB and CD, would be more remote
from

14 *The Elements of Euclid.*

‘ from each other towards the points E & F;
 ‘ than towards G and H; which would be con-
 ‘ trary to the definition of Parallels, where tis
 ‘ said, they are such as always keep the same
 ‘ distance, measur’d by perpendiculars.

12. Two right lines cannot enclose any space;
 that is to say, they cannot encompass it on all
 sides.

13. Two right lines cannot have one com-
 mon segment.



‘ By which I mean, that two
 ‘ right lines, *suppose* A B, and
 ‘ C B, meeting at the point B,
 ‘ cannot together make one fold
 ‘ line B D; but cutting one an-
 ‘ other separate again immedi-
 ‘ ately after their rencounter. For, if you de-
 ‘ scribe a circle from the point B as a center,
 ‘ A F D will be a semicircle, because the right
 ‘ line A B D, passing through the center B, will
 ‘ divide the circle into two equal parts. The
 ‘ segment C F D will be also a semicircle, be-
 ‘ cause C B D will be also a right line, and will
 ‘ pass through the center B: therefore the seg-
 ‘ ment C F D will be equal to the segment A F D,
 ‘ the part to the whole; which is repugnant
 ‘ to the ninth Maxime.

Adver

ADVERTISEMENT.

‘ **T**HERE are two sorts of Propositions: In
 ‘ some we have nothing but the bare Spe-
 ‘ culation of a Truth, without descending to
 ‘ Practice, which we call *Theorems*; in the
 ‘ other something is propos’d to be done, and
 ‘ those are call’d *Problems*.

‘ The first number of the quotations denotes
 ‘ the Propositions, the second the Book. As
 ‘ by the 2. of the 3. that is, by the second Pro-
 ‘ position of the third Book: but if only one
 ‘ number occur, it signifies such a Propoposi-
 ‘ tion of the book you are then upon.

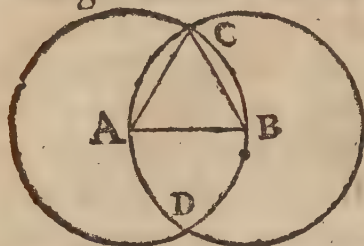
PROPOSITION I.

A PROBLEM.

*To draw an Equilateral Triangle upon
 any line given.*

LET the line AB be propos’d for the base
 of an Equilateral Triangle; from the
 center A at the distance AB describe the cir-
 cle BCD; and likewise from the center B at
 the distance BA describe the circle DAC cut-
 ting

ring the former at the point C. Then draw the lines AC and BC ; and all the sides of the triangle ABC will be equal.



Demonstration.

The Lines AB, and AC being drawn from the same center A to the circumference of the circle BCD are equal, by the Definition of a Circle ; the lines BA and BC are likewise equal being drawn from the center B to the circumference of the circle CAD. Lastly the lines AC and BC being equal to the same line AB, are also equal between themselves. All the three sides therefore of the triangle ABC are equal.



The USE.

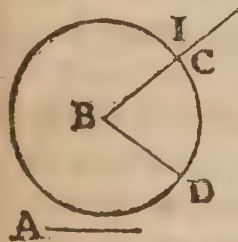
'The design of *Euclid* in
'placing this Problem here
'was only to demonstrate the
'two following Propositions
'But it may be also further
'serviceable for the measuring
'an inaccessible line, as for example, the line
'AB, which by reason of a River or Precipice
'cannot be approach'd. In such a case make
'a small Equilateral Triangle BDE, either of
'wood or copper, or the like ; and having
'placed it Horizontally upon B, observe the
point

‘ point A, by the side BD, and any other point
 ‘ C, by the side BE. Then transfer your Tri-
 ‘ angle along the line BC, and place it upon
 ‘ divers parts of the same line; till at length
 ‘ you find a point C, upon which placing the
 ‘ Triangle you shall see the point B, by the side
 ‘ CG, and the point A by the side CF. I say
 ‘ the lines CB and CA are equal; so that by
 ‘ measuring the line BC, you may know the
 ‘ line AB. I might further demonstrate that
 ‘ the lines AB, and BC are equal; but let it
 ‘ suffice that in this Proposition, you are taught
 ‘ the way of making an Instrument proper to
 ‘ take the dimensions of an inaccessible line.

PROPOSITION II.

A PROBLEM.

*From a point given to draw a line equal to
 another line given.*



LET the point propos'd be
 B, from which a line is to
 be drawn equal to the line A.
 Take with the Compass the
 length of the line A, and at that
 interval, making B the Center,
 describe the circle CD. Drawing then from
 the

the point B to which side you please, a line BI, or BD, 'tis evident it will be equal to the line A.

'*Euclid* proposes a more mysterious and intricate method of demonstrating this Proposition; but in practice we alwayes make use of this; in as much as, having taken with the compass the line A, 'tis as easy describing a circle from the center B, as from the center A.

PROPOSITION III.

A PROBLEM.

From a greater line to take a part equal to a less.

Suppose you were to take from the line BC, a part BI, equal to the line A. Take betwixt the points of the compass the length of the line A, and at that distance, from the center B describe a circle, which shall cut the line BC at the point I. 'Tis certain the lines BI, and A, are equal.

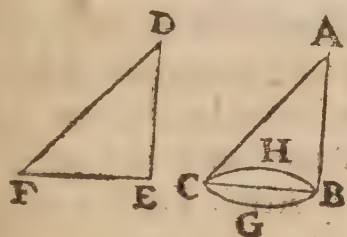
'The Use of these two preceding Propositions is sufficiently evident; for as much as we are frequently oblig'd in *practical Geometry* to draw one line equal to another; and to take a part of a greater line equal to a line that is less.

PROP.

PROPOSITION IV.

A THEOREM.

If two Triangles have two sides equal, each to the other respectively, and the angles also, form'd by those two sides, equal; their bases and other angles will be equal.



LET the triangles ABC, DEF , have two sides equal each to the other respectively; that is to say, let AB be equal to DE , and AC to DF ; and let the angles BAC, EDF , form'd by those sides be also equal: I say, the bases BC, EF , are equal, and the angles ABC, DEF ; ACB, DFE , are equal; and lastly, the whole triangles equal in all respects.

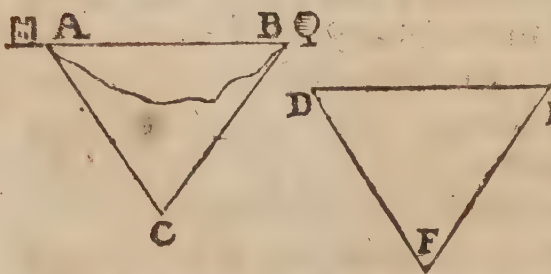
Demonstration.

Suppose the triangle DEF to be plac'd upon the triangle ABC : the side DE being upon AB , they will not exceed each other, because they are suppos'd to be equal; so that the point E will be upon B , and the point D upon the point A . For the same reason the line DF will fall upon AC . For if it should fall on the outside

side of it, the angle EDF would be greater than the angle BAC; and if it should fall within AC, the angle EDF would be less: and yet they are suppos'd to be equal. Therefore since the point D is upon the point A, and the line DF falls upon the line AC, to which it is equal, they will not exceed each other, but the point F will fall upon C. Lastly, since the points E and F of the line EF, fall upon B and C; the line EF will fall upon BC; because it can neither fall higher as in BHC, nor lower as in BGE; for then two right lines would enclose space; which is contrary to the twelfth Maximæ. Therefore the two triangles do not at all exceed each other; but not only the basess BC, EF, but also the angles ABC, DEF; and ACB, DFE, are equal.

Coroll. An Equilateral triangle hath all its angles equal.

The USE.



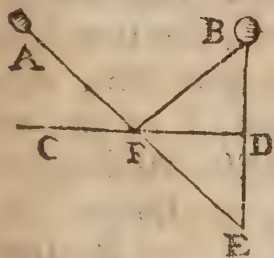
' Suppose II
' was to mea-
' sure an in-
' accessible
' line AB. II
' observe

' from the point C the points A, and B; and
' then measure the angle C. This done, placing
' a board horizontally, and observing successive-

' ly

ly by a rule the points A and B, I draw two lines according to the rule, which make the angle C; and measure with a yard the lines AC, and BC, which are suppos'd accessible. Then going into an open field, and placing my board again horizontally upon the point F, and observing the lines that I drew upon it, I make an angle DFE equal to the angle C. I make likewise FD, FE, equal to CA, CB. Then according to this proposition the lines AB, and DE, are equal. So that measuring by the yard the accessible line DE, I shall know AB, which is inaccessible.

Another USE.



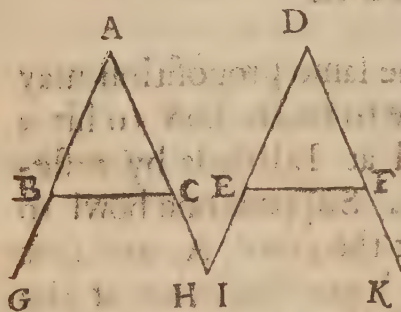
The same proposition may serve to teach how to hit a bowl at Billiards by reflexion. Suppose one bowl to be at the point A, and that which you would hit at the point B, and CD the Billiard table. Imagine then a perpendicular BDE, and take the line DE equal to BD. I say, if you direct the bowl from the point A to E, the reflexion will carry it to B. For in the triangles BFD, EFD, the side FD being common, and the sides BD and DE equal; the angles BFD, EFD are equal, by this proposition. The angles

Angles AFC, DFE, being opposite, are also equal;
 as I shall demonstrate hereafter. Therefore
 the angle of Incidence AFC, is equal to the
 angle of Reflexion BFD; and by consequence
 the Reflexion will be by AFB.

PROPOSITION V.

THEOREM.

*In Isosceles, or Equicrural triangles, the angles
 that are above the Base are equal: as also
 those that are below it.*



Let the Isosceles be
 ABC, that is to say
 let the sides AB, AC
 be equal. I say the an-
 gles ABC, ACB are
 equal; as also the an-
 gles GBC, HCB, that
 are below the base BC. Suppose another tri-
 angle DEF, having the angle D equal to the
 angle A; and the sides DE, DF, equal to AB
 AC. Since the sides AB, AC are equal, all the
 four lines AB, AC, DE, DF will be equal.

Demonstration. Since the sides AB, DE; AC,
 DF, are equal; as also the angles A, and D
 if the triangle DEF be plac'd upon ABC, the
 will

will not exceed each other, but the line DE will fall upon AB; DF upon AG; and EF upon BC (by the 4th.) therefore the angle DEF, will be equal to ABC. And because one part of the line DE falls upon AB, the whole line DI will be upon AG; otherwise two right lines would have a common segment; therefore the angle IEF will be equal to GBC. Suppose then the triangle DEF turn'd, and apply'd another way to the triangle ABC, that is to say, so as DF may fall upon AB, and DE upon AC. Since the four lines AB, DF; AC, DE, are equal; as also the angles A and D: the triangles will likewise agree this way, and the angles ACB, DEF; HCB, IEF, will be equal. Now by the first comparing them it appear'd, that the angle ABC was equal to the angle DEF; GBC to IEF: therefore the angles ABC, ACB being equal to the same DEF; and GBC, HCB, also equal to the same IEF, they are equal among themselves.

'I was unwilling to make use of *Euclid's* demonstration, because being very difficult, it might discourage beginners.

D 2

PROP.

PROPOSITION VI.

THEOREM.

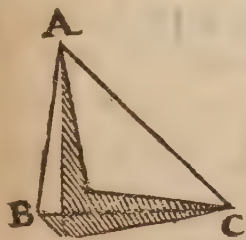
If two angles of a triangle be equal, the triangle will be an Isosceles.

LET the angles ABC , ACB of the triangle ABC be equal: (*see Fig. preced.*) I say it is an Isosceles; that is to say, the two sides AB , AC , which are opposite to the equal angles, are equal. Suppose the triangle DEF to have a base EF equal to BC , and the angles DEF equal to ABC , as also DFE equal to ACB : since the angles ABC , ACB are suppos'd to be equal, all the four angles ABC , ACB , DEF , DFE , will be equal. Suppose again therefore the base EF to be plac'd upon the base BC , so that the point E lie upon the point B , the bases being suppos'd equal it is evident they will not exceed each other. Further, the angle E being equal to the angle B , and the angle F to the angle C ; the line ED will fall upon the line BA , and FD upon CA : so that the lines ED and FD will meet at the point A . From whence it follows, that the line ED is equal to BA .

Let then the triangle DEF be turn'd to the other side, and be applied another way to the tri-

triangle ABC: that is to say, so that the point E lie upon C, and F upon B: the bases BC, FE will perfectly agree, being suppos'd to be equal: and because the angles F, and B; E, and C, are also suppos'd to be equal, the side FD will fall upon BA, and ED upon CA; and the point D upon A. Therefore the lines AC, DE will be equal. Whence it follows, that the sides AC, AB are equal between themselves, being equal to the same side DE.

The USE.



' This Proposition may serve
' for taking the dimensions of
' any sort of inaccessible lines.
' 'Tis said that *Thales* was the
' first that measur'd the height
' of Obelisks by their sha-
' dows: it may be done by this Proposition.
' For if you were to measure the height of the
' Obelisk AB; do but expect till the Sun be
' elevated 45 degrees above the Horizon; that
' is to say, till the angle ACB be 45 degrees;
' and, by the sixth Proposition, the shadow BC
' will be equal to the Obelisk AB. For since
' the angle ABC is a right angle, and the angle
' ACB half a right one, or of 45 degrees; the
' angle CAB will be half a right one, as I shall
' prove hereafter. Therefore the angles BCA,
' BAC,

‘BAC, are equal: and (*by the 6.*) the sides AB,
 ‘BC, are also equal. I can also measure the same
 ‘height without making use of the shadow, by
 ‘taking a stand so far from the point B, as that
 ‘the angle ACB may be half a right angle,
 ‘which may be known by a *Quadrant*.

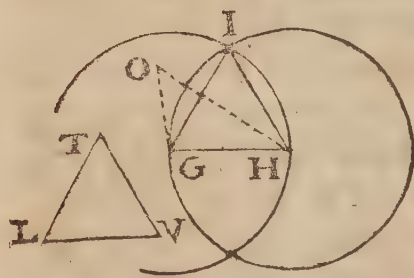
‘These Propositions are of frequent use in
 ‘Trigonometry, and in all other tracts,

‘The *seventh* Proposition may be omitted,
 ‘because tis of no other use but to demonstrate
 ‘the *eighth*, which may be done without it.

PROPOSITION VIII.

THEOREM.

*If two Triangles have all their sides equal, their
 opposite angles will also be equal.*



LET the side
 GI be equal to
 LT; HI, to VT;
 GH, to LV; I say,
 that the angle GIH,
 will be equal to the
 angle LTV; IGH,
 to the angle L; and IHG, to the angle V.
 From the center H, at the distance HI, describe
 the circle IG; and from the center G, at the
 distance GI, the circle HI.

Demon-

Demonstration.

Suppose the line LV brought upon HG: they would not exceed each other, because they are suppos'd to be equal. I add, that the point T will fall precisely upon the point I: For it ought to reach precisely to the circumference of the circle IG, because by the supposition the lines HI and VT are equal. It ought in like manner to reach to the circumference of the circle IH, because the lines GI and LT are equal. So then it will light upon the point I, being the point where those two circles cut each other. Indeed if it fell any where else, as upon O, the line HO, that is to say VT, would be greater than HI; and the line GO, that is LT, would be less than GI; which is against the supposition. Whence I conclude, that the triangles will exactly correspond, and the angle GIH be equal to the angle LTV.

The Use.

' This Proposition is necessary for the proof
' of those that follow. And further, when we
' cannot take the measure of an angle, because,
' the lines meeting in a solid body, we cannot
' apply our Instruments to it; we must take
' the three sides of the triangle, and make an-
' other upon a paper, whose angles we may
' measure. This is a very ordinary practice in

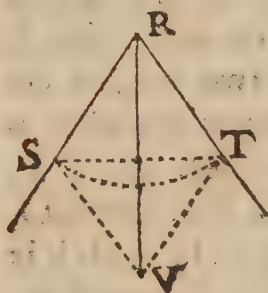
Gro.

‘ *Gnomonicks*, or Dialling ; and in the treatises
 ‘ concerning cutting pretious stones, so as to fit
 ‘ the pannels, and to retain the waters.

PROPOSITION IX.

PROBLEM.

To divide an Angle into two equal parts.



LET the angle SRT be propos'd to be divided into two equal parts. Take the Compass, and from the center R , at any distance, draw the arch ST , cutting off two equal lines RS, RT . Then draw the right line ST , and (*by the 1.*) describe an equilateral triangle STV . I say, the line VR divides the angle into two equal parts: that is to say, the angles VRT , and VRS , are equal.

Demonstration.

The triangles VRS , and VRT , have the side VR common ; and the side RT was taken equal to the side RS : the base also SV , is equal to VT , because the triangle SVT is equilateral. Wherefore (*by the 8.*) the angles SRV , VRT , are equal.

The

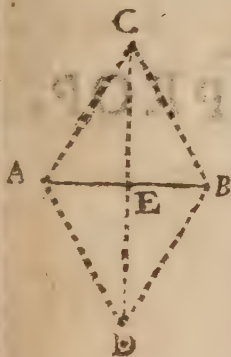
The USE.

This proposition is very useful to divide the fourth part of a circle into degrees : for tis the same thing to divide an arch, as an angle into two equal parts ; and the line RV does both, that is, it divides both the arch ST, and the angle SRT. Having therefore apply'd the semidiameter to the fourth part of a circle, you cut off an arch of 60 degrees, which divided equally gives an arch of 30 ; and that again divided, makes one of 15 degrees. Tis true, to finish this division, we must divide an arch into three equal parts, but that is not to be done Geometrically. Pilots also divide the Compass into 32 winds by the help of this Proposition only.

PROPOSITION X.

A PROBLEM.

To divide a right line into two equal parts.



Suppose the line AB was to be divided into two equal parts ; upon the line AB describe an equilateral triangle ABC, (*by the 1.*) and divide the angle ACB into two equal parts by the line DC, (*by the 9.*) I say the

the line AB is divided equally at the point E, that is to say, the lines AE and EB are equal.

Demonstration.

The triangles ACE, and BCE have the side CE common, and the sides CA and CB are equal, because the triangle ACB is equilateral; and the angle ACB being divided equally, the angles ACE and BCE are also equal. Therefore (by the 4.) the bases AE and BE are equal.

The Use.

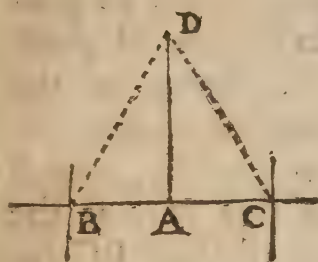
Great use is made of this Proposition, ordinary practices frequently requiring us to divide a line in the middle, which Geometricians require should be done exactly at the first dash, by a method that is infallible, and not by essays. This practice is likewise principally useful for dividing measures into less parts.

PROP.

PROPOSITION XI.

A PROBLEM.

To draw a Perpendicular to a line given, upon a point of the same line.



Suppose you were to raise a perpendicular upon the point A of the line BC. Take two equal lines AB and AC on both sides the point A, and make an equilateral triangle BDC upon the line BC, (*by the 1.*) I say the line AD is perpendicular, that is to say, the angles BAD and CAD are equal.

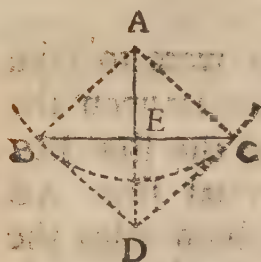
Demonstration.

The triangles BAD, and CAD have the side AD common, the sides AC and AB are equal, and the bases BD and DC also equal: therefore (*by the 8.*) the angles BAD, and CAD, are equal; and (*by the 10. def.*) the line AD perpendicular to BC.

PROPOSITION XII.

A PROBLEM.

To draw a perpendicular to a line given, from a point, which is out of the line.



IF you would draw a perpendicular to the line BC, from the point A : having set the foot of the compass upon A, describe the circle BC, which shall cut the line BC, at the points B and C. Then divide the line BC into two equal parts at the point E. I say the line AE is perpendicular to BC. Draw the lines AB, AC.

Demonstration. The triangles BEA, and CEA, have the side AE common ; and the sides EC and EB equal, the line BC having been equally divided at the point E ; the bases AB and AC, being drawn from the center A to the circumference BC, are likewise equal : therefore the angles AEB, and AEC, are equal, (*by the 8.*) and the line AE perpendicular, (*by defin. 10.*)

The method, in practice, of dividing the line BC in the middle, is to describe two arches at D, at the same interval, from the centers B and C.

The

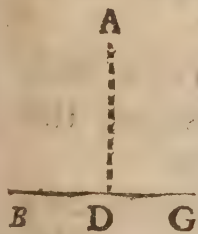
The USE.

‘We have need of a Plummer, or Squaring-line almost in all our operations: no angles are of use in buildings but the right; and all chairs, benches, tables, buffers, and other moveables, are fram’d by the square. No survey of Land can be taken without making use of perpendicular lines; nor can *Dialing* be perform’d without them. The Carpenter’s Level contains a right angle, and the same is preferr’d before any other, especially by the *French*, in Fortifications. Lastly, not only *Mathematicians*, but also the greatest part of practical Artisans, require that we should know how to draw a perpendicular.

PROPOSITION XIII.

A THEOREM.

One line falling upon another makes with it either two right angles, or two angles equal to two right ones.



L Et the line AD fall upon BG; I say, it will make with it either two right angles; or two angles, one obtuse, and the other acute, which joyn’d together shall be of equal value with two right ones.

Demon.

Demonstration.

Suppose the line AD to fall perpendicularly upon BG, then tis evident (*by defin. 10.*) that the angles ADB, and ADG, are equal, and by consequence right angles. Or,



Secondly, suppose the line ED not to fall perpendicularly upon BC, and draw a perpendicular AD (*by the 11.*) the angles ADB, and ADC are right angles, which are of equal value with the three angles ADC, ADE, EDB. But the obtuse angle EDC, and the acute angle EDB, are of equal value with the three angles ADC, ADE, and EDB: therefore the angles EDC, and EDB, are of equal value with two right ones.

This Proposition may be more easily demonstrated by describing a semicircle from the center D vpon the line BC. For the angles EDB, and EDC, will require a semicircle for their measure, which is the measure of two right angles, as I have shown before; in the 8. definition.

Corollary 1. If the line AD falling upon BC, make one right angle ADC; it is evident the other, ADB, will be also a right angle.

Coroll. 2. If the line ED, falling upon BC, make the angle EDB acute; the angle EDC will be obtuse.

The

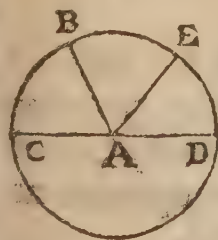
The USE.

By this means, when we know one of the angles which is made by one line falling upon another, we know also the other: as for example, if the angle EDB be one of 70 degrees, taking away seventy from 180, there will remain 110 for the angle EDC. This operation does frequently occur in *Trigonometry*; and also in *Astronomy*, for finding the eccentricity of the circle through which the Sun annually passes.

PROPOSITION XIV.

A THEOREM.

If two lines meeting together at the same point of another line, make with it two angles equal to two right ones; they will make but one and the same line.



Suppose the lines CA, and DA, to meet at the point A of the line AB; and that the angles adjoining, CAB, and BAD, are equal to two right ones. I say, the lines CA and DA are but one and the same line; so that CA being continued, will fall precisely upon AD.

Imagine

Imagine, if you please, that CA continu'd will pass on to E, and from the center A describe a circle.

Demonstration.

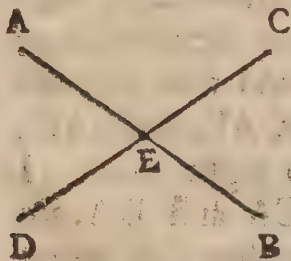
If you say that CAE is a right line, the arch CBE will be a semicircle. But 'tis suppos'd, that the angles CAB, and BAD are equal to two right ones, and that therefore their measure is a semicircle. Therefore the arches CBE, and CBD will be equal; which is impossible, one being a part of the other. Therefore the line CA being continu'd, will make but one and the same line with AD.

PROPOSITION XV.

A THEOREM.

*If two right lines cut each other, the opposite angles * at the top will be equal.*

Ἐν κορυφῇ. Eucl. au sommet. Gall.



LET the lines AB and CD cut each other at the point E: I say, the angles AEC, and DEB, which are opposite at the top, are equal.

Demonstration,

The line CE falling upon the line AB, makes the

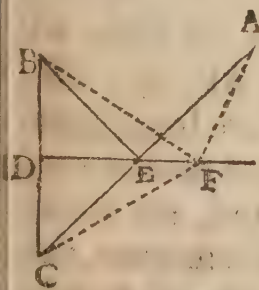
the angles AEC and CEB equal to two right ones, (*by the 13.*) In like manner the line BE falling upon the line CD, makes the angle CEB and BED equal to two right ones. Therefore the angles AEC, CEB, taken together, are equal to the angles CEB, BED; therefore taking away the angle CEB from both, the angle AEC will remain equal to DEB, (*by the 3. Maxime.*)

Coroll. 1. If two lines DE, and EC, concurring at the same point E of the line AB, form with it the opposite angles AEC, DEB equal, DE and EC make but one right line.

Demonstration.

The line EC falling upon the line AB, makes the angles AEC, and CEB equal to two right ones, (*by the 13.*) 'Tis suppos'd likewise that the angle DEB is equal to the angle AEC. Therefore the angles DEB, BEC, are equal to two right ones. And (*by the 14.*) the lines CE and ED make but one right line.

The use.



' The two preceding Propositions are made use of to prove, that two lines make but one total. As for example, in Catoptricks or Perspectives, where that is required to prove, that of all the lines
F that

' that can be drawn by reflexion from the point
 ' A to the point B, those are the shortest
 ' which make the angle of Incidence equal to
 ' the angle of Reflexion. As for example; if the
 ' angles BED and AEF be equal, the lines AE
 ' and EB, are shorter than AF, and FB. From
 ' the point B draw a perpendicular BD, and
 ' make the lines BD and CD equal; then draw
 ' EC, and FC. First in the triangles BED and
 ' CED the side DE is common; and the sides
 ' BD, and DC being equal, as also the angles
 ' BDE, and CDE; the bases BE, and CE will
 ' be equal; as also the angles BED, and DEC
 ' (*by the 4.*) In like manner I may prove, that
 ' BF, and CF are equal.

Demonstration.

' The angles BED and DEC are equal, and the
 ' angles BED and AEF are suppos'd likewise to
 ' be equal; therefore the opposite angles DEC
 ' and AEF will be equal; and (*by the Coroll. of
 ' the 15.*) AEC one right line; and by con-
 ' sequence AFC is a triangle, of which the sides
 ' AF and FC must be longer than AEC, that is
 ' to say, than AE, and EB. But the lines AE
 ' and FC are equal to the lines AF, and FB
 ' therefore the lines AF and FB are longer than
 ' the lines AE and EB. And since nature
 ' ses always act by the shortest lines, the Reflexion
 ' will always happen in such a manner, that
 ' the angles of Reflexion and Incidence may be
 ' equal.

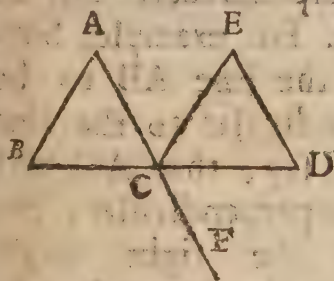
Fur

Further, because we can easily prove, that all the angles that can be made upon a plane about the same point, are equal to four right angles; for as much as in the first figure of this proposition, the angles AEC and AED are equal to two right ones, as also BEC and BED to two more: we make a general rule to determine what Polygons may be joyn'd in paving a Hall. Accordingly we say, that four squares, six triangles, and three hexagones, may be used for that purpose; and that therefore Bees are always observ'd to make their little cells of the last, that is, of figures consisting of six sides.

PROPOSITION XVI.

A THEOREM.

The external angle of a triangle is greater than either of the internal opposite angles.



Produce the side BC of the triangle ABC: I say the external angle ACD, is greater than either of the internal opposite angles, ABC, or BAC. Suppose the triangle ABC to be mov'd along the line BD,
 Fig 2

and carry'd into the place of CED.

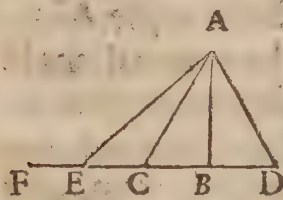
Demonstration.

Tis impossible that the triangle ABC should be so mov'd, but the point A must come into the place of the point E; and then twill appear, that the angle ECD, that is to say, ABC, is less than the angle ACD: therefore the internal angle ABC is less than the external ACD.

Tis likewise easie to prove, that the angle A is less than the external angle ACD: for having prolong'd the side AC as far as F, the opposite angles BCF, and ACD, are equal (*by the 15.*) Therefore causing the triangle ABC to slide along the line ACF, I shall demonstrate, the angle BCF to be greater than the angle A.

The USE.

We may draw from this proposition many most useful conclusions. As first, that from a point given only one perpendicular can be drawn to the same line. For example, Suppose



pose the line AB to be perpendicular to the line BC: I say, that AC will not be perpendicular; because the right angle ABD must be greater than the internal angle ACB; therefore ACB cannot be a right angle, nor AC a perpendicular. See

‘ Secondly, that from the same point A cannot be drawn more than two equal lines; for example, AC, and AD; and if you draw a third as AE, it will not be equal to the former. For since AC and AD are equal, the angles ACD, and ADC, are equal, (*by the 5.*) but in the triangle AEC, the external angle ACB is greater than the internal AEC: and therefore likewise the angle ADE, is greater than the angle AED. Therefore the lines AE, and AD, are not equal; nor by consequence AC and AE.

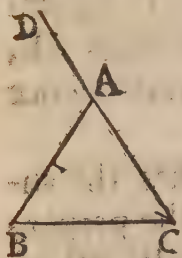
‘ Thirdly, that if the line AC makes the angle ACB acute, and ACF obtuse, the perpendicular drawn from the point A will fall on the side of the acute angle. For if you say that AE is a perpendicular, and that AEF is a right angle; the right angle AEF would be greater than the obtuse ACE. These conclusions are serviceable for measuring Parallelograms, Triangles, and Trapezia, and to reduce them into rectangular figures.

P R O P.

PROPOSITION XVII.

A THEOREM.

Any two angles of a triangle are less than two right ones.



LET the triangle be ABC ; I say, that any two of its angles taken together, as BAC , and BCA , are less than two right ones. Produce the side CA to the point D .

Demonstration.

The internal angle C , is less than the external BAD , (*by the 16.*) Add therefore to both the angle BAC ; the angles BAC , and BCA , will be less than the angles BAC , and BAD ; yet those are but equal to two right ones, (*by the 13.*) therefore the angles BAC , and BCA , are less than two right ones.

After the same manner I can demonstrate the angles ABC , and ACB , to be less than two right ones, by producing the side BC .

Coroll. If one angle of a triangle be a right, or obtuse, angle, the others are acute.

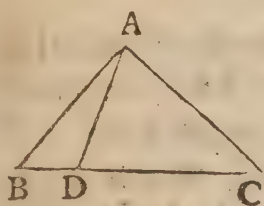
“This Proposition is necessary to demonstrate those that follow.

PROP.

PROPOSITION XVIII.

A THEOREM.

In every triangle whatsoever the greatest side is oppos'd to the greatest angle.



Suppose the side BC of the triangle ABC, to be greater than the side AC: I say, the angle BAC, that is oppos'd to the side BC, is greater than the angle B, which is oppos'd to the side AC. Cut the line BC in D, so that CD may be equal to AC; then draw the line AD.

Demonstration.

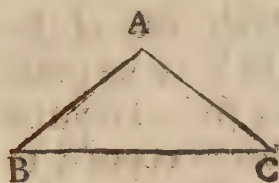
Since the sides AC, and CD, are equal, the triangle ACD will be an *Isoceles*, and (by the 5.) the angles CDA, and CAD, equal. Now the whole angle BAC is greater than the angle CAD: therefore the angle BAC is greater than the angle CDA; which yet, being an external angle in respect of the triangle ABD, is greater than the internal B, (by the 16.) Therefore the angle BAC is greater than the angle B.

PROP.

PROPOSITION XIX.

A THEOREM.

In every triangle the greatest angle is oppos'd to the greatest side.



LET the angle A of the triangle BAC, be greater than the angle ABC. I say, the side BC, which is oppos'd to the angle A, is greater than the side AC, that is oppos'd to the angle B.

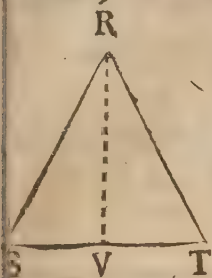
Demonstration.

If the side BC be not greater than the side AC, tis either equal; and then the angles A and B would be equal, (*by the 5.*) which is contrary to the supposition: or less; and if so, the side AC being greater than BC, the angle B would be greater than the angle A, though the contrary be suppos'd. It remains therefore that the side BC be greater than the side AC.

The U S E.

'We may prove from these propositions,
'not only that no more than one perpendicular
can

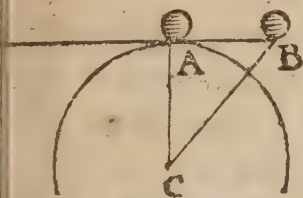
can be drawn from the same point to the same line; but also that it is the shortest of all. As



‘for example; if the line RV
‘be perpendicular to ST, it will
‘be less than RS: because the
‘angle RVS being a right angle,
‘the angle RSV will be an acute,
‘(by the Coroll. of the 17.) and
‘the line RV will be less than

RS, (by the preceding.) Therefore Geometricians do always make use of a perpendicular, when they take the dimensions of any thing, and reduce irregular figures to such as have one or more right angles. I add, that it being impossible that more than three perpendiculars should meet at the same point, it cannot be imagin’d that there should be more than three species or kinds of quantity, a line, a superficies, and a solid body.

‘By these propositions we likewise prove, that a bowl exactly round cannot rest, but upon such a certain point. For example; let the



‘line AB represent a plane,
‘and C the center of the
‘earth, and that CA be
‘drawn perpendicular to
‘the line AB; I say, that a
‘bowl being plac’d upon

the point B, cannot rest there. For a heavy body cannot rest, when it may descend. Now

G

the

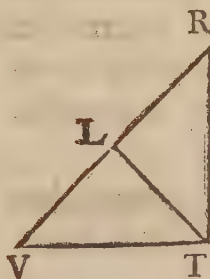
‘ the bowl B moving towards A continually descends, and approaches nearer the center of the earth C; because in the triangle CAB, the perpendicular CA is shorter than BC.

‘ In like manner we prove, that a liquid body must flow from B to A, and that its superficies must be round.

PROPOSITION XX.

A THEOREM.

Any two sides of a triangle taken together are greater than the third.



I say that the two sides TL, LV are greater than the side TV. Some men prove this Proposition by the definition of a right line which is the shortest that can be drawn from one point to another; therefore the line TV, is less than the two lines TL and LV.

But it may also be demonstrated another way. Continue the side VL to R, so that the lines LR, and LT be equal; then draw the line RT.

Demonstration.

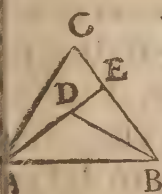
The sides LT, and LR, of the triangle LTR are equal; therefore the angles R, and LTR are equal, (*by the 5.*) But the angle RTV is greater

greater than the angle RTL : therefore the angle RTV is greater than the angle R : and (by the 19.) in the triangle RTV , the side RV , that is to say, the sides LT and LV , are greater than the side TV .

PROPOSITION XXI.

A THEOREM.

If a small triangle be describ'd within a greater, upon the same base, the sides of the small one will be less than those of the greater; but they will form a greater angle.



LET the small triangle ADB be describ'd within the triangle ACB , upon the same base AB . I say first, the sides AC and BC are greater than the sides AD and BD .

Continue the line AD to E .

Demonstration.

In the triangle ACE , the sides AC and CE , are greater than the side AE alone, (by the 20.) Therefore adding to them the side EB ; the sides AC , and CEB , are greater than the sides AE , and EB . In like manner in the triangle DBE , the two sides BE and ED are greater than the side BD alone, and adding the side AD ,

the sides ADE, and EB, will be greater than AA and BD.

I say further, that the angle ADB is greater than the angle ACB: for the angle ADB is an external angle in respect of the triangle DBE and therefore greater than the internal DEE (by the 16.) In like manner the angle DEE being an external angle in respect of the triangle ACE, is greater than the angle ACE therefore the angle ADB is greater than the angle ACB.

The U S E.

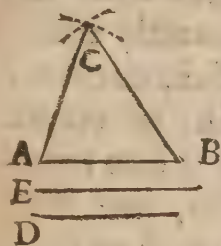
‘By the help of this proposition we demonstrate in *Opticks*, that the Base AB viewed from the point C, will appear less, than when it is beheld from the point D; according to that principle, That quantities view’d under a greater angle will appear greater. Therefore ‘tis, that *Vitruvius* advises, not much to lessen the tops of very high Pillars, because they being so remote from our sight, quickly appear slender enough without being diminished.

P R O P

PROPOSITION XXII.

A THEOREM.

To describe a triangle, whose sides shall be equal to three sides given, provided that any two of them be greater than the third.



LET it be propos'd to describe a triangle, whose sides shall be equal to three lines given, AB, D, and E. Measure with the compass the line D, and setting one foot thereof upon the point B make an arch. Then take the line E, and placing the foot of your compass upon the point A make another arch, cutting the former at the point C. Which done draw the lines AC and BC. I say that the triangle ABC, is such a one as you desire.

Demonstration.

The side AC is equal to the line E, because it reaches to the arch, which is drawn from the center A at the distance of the line E; and for the same reason the side BC is equal to the line D: therefore the three sides AC, BC, and AB, are equal to the lines E, D, and AB.

I added a Proviso, that the two lines should be greater than the third: because otherwise if the

the lines D and E were less than the line AB, the arches could not cut each other.

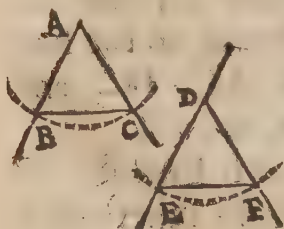
The 11 S E.

‘ This Proposition may be useful for describing a figure equal or like to another : for having divided that , which is propos’d to be ‘ equall’d or imitated, into triangles ; and made ‘ other triangles, having equal sides with the ‘ former ; we shall have a figure exactly equal. ‘ But if we desire only one that is like, but less ; ‘ as when we would describe a plain, or country upon paper : having divided it into triangles, and measur’d all their sides, we must ‘ make similar triangles ; giving to each of their ‘ sides so many parts of a *Scale*, or line divided ‘ into equal parts, as the sides of the triangles ‘ propos’d have of yards or feet.

PROPOSITION XXIII.

A PROBLEM.

To make an angle equal to another at a point of a line given.



Suppose you were to make an angle at the point A of the line AB, equal to the angle EDF. Describe from the points A and D as centers

two

two arches BC, and EF, at the same wideness of the compass; then take the distance EF, and having measur'd as much at BC, draw the line AC. I say the angles BAC, and EDF, are equal.

Demonstration.

The triangles ABC, and DEF, have the sides AB, and AC, equal to the sides DE, and DF; since the arches BC and EF were describ'd with the same wideness of the compass: the bases also BC and EF are equal, therefore the angles BAC and EDF are equal, (*by the 8.*)

The U S E.

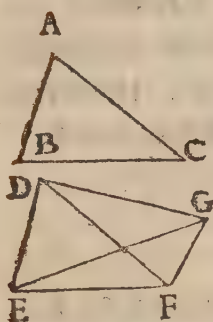
This Problem is so necessary in *Geodesia*, *Fortifications*, *Perspective*, *Dialling*, and all other parts of the Mathematicks, that the greatest part of their Operations would be impossible, if we did not know how to make one angle equal to another, or of such a number of degrees as we please.

PROP.

PROPOSITION XXIV.

A THEOREM.

Of two triangles, having each two sides equal to two of the other, that which has the greatest angle, has also the greatest base.



LET the triangles ABC, DEF, have the sides AB and DE; AC and DF equal; and let the angle BAC be greater than the angle EDF. I say, the base BC is greater than the base EF.

Make the angle EDG equal to the angle BAC, (*by the 23.*) and the line DG equal to AC; then draw the line EG. First the triangles ABC and EDG, having the sides AB and DE, AC and DG, equal, and the angle EDG equal to the angle BAC; their bases BC and EG will be equal (*by the 4.*) and the lines DG and DF being both equal to AC, will be equal betwixt themselves.

Demonstration.

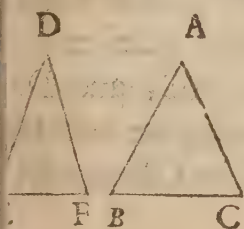
In the triangle DGF, the sides DG and DF being equal, the angles DGF and DFG will be equal, (*by the 5.*) But the angle EGF is less than the angle DGF, and the angle EFG is greater than the angle DFG. Therefore in the

the triangle EFG, the angle EFG will be greater than the angle EGF: and therefore (*by the 18.*) the line EG oppos'd to the greater angle EFG, will be greater than EF. Therefore BC, being equal to EG, is greater than EF.

PROPOSITION XXV.

A THEOREM.

Of two triangles, having each two sides equal to two of the other, that which has the greatest base, has likewise the greatest angle.



LET the two triangles ABC, DEF, have the sides AB, DE; and AC, DF, equal; and let the base BC be greater than the base EF. I say, that the angle A will be greater than the angle D.

Demonstration.

If the angle A be not greater than the angle D; it will be either equal, and then the bases BC, EF, will be equal, (*by the 4.*) or it will be less, and the base EF greater than the base BC, (*by the 24.*) but both are contrary to the supposition.

“ These Propositions are necessary to demonstrate those that come after.

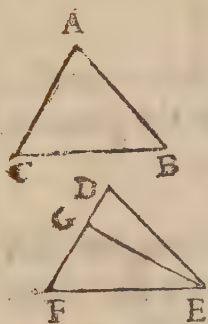
H

PROP

PROPOSITION XXVI.

A THEOREM.

If one triangle has one side, and two angles, equal to those of another triangle; tis equal to it in all respects.



LET the angles ABC , DEF , ACB , DFE , of the triangles ABC , DEF , be equal; and the sides BC , and EF , which are between those angles, also equal. I say, that the other sides are equal; for example, AC , and DE . Imagine, if you please, the side DE to be greater than AC ; and cutting GF equal to AC draw the line GE .

Demonstration.

The triangles ABC , GEF , have the sides BC ; AC , GE , equal; the angle C is also supposed to be equal to F . Therefore (*by the 4.*) the triangles ABC , GEF , are equal in all respects and the angles GEF , and ABC , are equal. But we supposed the angles ABC , DEF , to be equal and so, the angles DEF , GEF , would be equal; that is, the whole to the part; which is impossible. Therefore the side DE will not be greater than the side AC , nor AC greater than DE .

D

DF, because the same demonstration may be made in the triangle ABC.

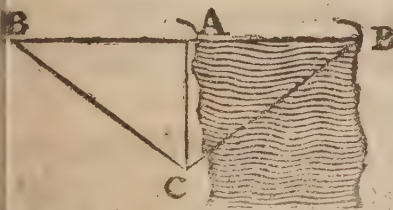
Again, suppose the angles A and D, C and F to be equal; and also the sides BC, and EF, oppos'd to the angles A and D, to be equal. I say, the other sides are equal. For if DF be greater than AC, cut GF equal to AC, and draw the line GE.

Demonstration.

The triangles ABC, GEF, having the sides EF, BC; FG, CA, equal, will (*by the 4.*) be equal in all respects; and the angles EGF, BAC, will be equal. But we suppos'd, that the angles A and D were equal, therefore the angles D, and EGF, must be equal, which is impossible, since the angle EGF, being the external angle in respect of the triangle EGD, must be greater than the internal D, (*by the 16.*) therefore the side DF is not greater than AC.

The USE.

'Thales made use of this proposition to measure inaccessible distances. For example: the



'distance AD being
'propos'd, he would
'draw from the point
'A, the line AC per-
'pendicular to AD:

then describing a semicircle at the point C, would measure the angle ACD, and take an-

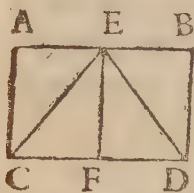
H 2

other

' other equal to it on the other side, prolonging
 ' the line CB till it met with the line DA at the
 ' point B; and then demonstrated the lines
 ' AD and AB to be equal; so that measuring
 ' the line AB, which was accessible, he could
 ' know the other which was not. For the two
 ' triangles ADC, and ABC, have the right an-
 ' gles CAD, and CAB equal, the angles ACD
 ' and ACB are also taken equal; and the side
 ' AC is common to both: therefore (*by the 26.*)
 ' the sides AD and AB are equal.

A L E M M A.

A line which is perpendicular to one of two parallel lines, is also perpendicular to the other.



' Let the parallel lines be AB, and
 ' CD, and let EF be perpendicular
 ' to CD. I say, tis also perpendicular
 ' to AB. Cut the line CF equal
 ' to FD, and upon the points
 ' C and D raise two perpendiculars to CD
 ' which, *by the definition of Parallels*, will be
 ' equal to FE; then draw the lines EC and ED

Demonstration.

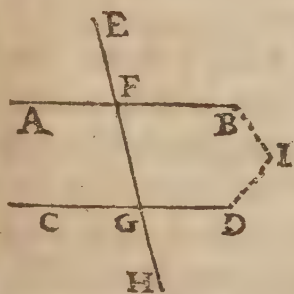
' The triangles CEF, and FED, have the
 ' side FE common, the sides CF, FD, equal
 ' and the angles CFE, and EFD, right, and by
 ' consequence equal; therefore (*by the 4.*) the
 bases

‘bases EC, ED, and the angles FED, FEC, and
 ‘FCE, FDE, will be equal; the two last of
 ‘which being taken away from the right angles
 ‘ACF and BDF, leave the two angles ACE,
 ‘and BDE, equal; therefore the triangles
 ‘CAE, DBE, will have (*by the 4.*) the angles
 ‘DEB, CEA, equal; which being added to the
 ‘equal angles CEF, FED, make the angles
 ‘FEB, and FEA, equal; therefore the line EF
 ‘is perpendicular to AB.

PROPOSITION XXVII.

A THEOREM

*If a line, falling upon two others, makes with them
 the alternate angles equal, those two lines are
 parallel.*



LET the line EH, falling
 upon the lines AB and
 CD, make with them the
 alternate angles AFG, FGD
 equal. I say first, the lines
 AB, and CD, will never
 concur, though continu'd as
 far as you please. For suppose them to concur
 in I, and that FBI, and GDI, are two right lines.

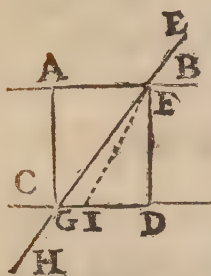
Demonstration.

If FBI, and GDI, be two right lines, FIG
 is

is a triangle ; and (*by the 16.*) the external angle AFG , is greater than the internal FGI . They cannot therefore be equal, if the lines AB and CD ever concur.

‘ But because we have examples of some
‘ crooked lines, which never concur ; and yet
‘ are not parallels, approaching still nearer and
‘ nearer to each other,

I say secondly, that if the line EH , falling upon the lines AB , and CD , makes the alternate angles AFG , and FGD equal ; the lines AB , CD , are parallel, or in all respects equally remote from each other, so that the perpendiculars between them will be equal. From the point G draw the perpendicular GA to the line AB ; and taking GD equal to AF , draw FD .



Demonstration.

The triangles AGF , and FGD , have the side FG common ; the side GD is also taken equal to the side AF , and the angles AFG and FGD suppos'd to be equal. Therefore (*by the 4.*) the bases AG and FD are equal, and the angle CDF is equal to the right angle CAB ; therefore FD is perpendicular. I add, that the line AB is parallel to CD : for the only parallel line that can be drawn from the point F to the line CD , ought to pass by the point A , according

cording to the definition of Parallels; which requires, that the perpendicular lines AG and FD be equal.

PROPOSITION XXVIII.

A THEOREM.

If a line, falling upon two others, makes the external angle, equal to the internal opposite angle on the same side ; or the two internal angles on the same side equal to two right ones ; those two lines will be parallel.

IN the precedent figure, suppose the line EH, falling upon AB, and CD, to make first the external angle EFB equal to the internal opposite angle on the same side FGD. I say, that the lines AB, and CD, are parallel.

Demonstration.

The angle EFB is equal to the angle AFG, being oppos'd to it at the top (*by the 15.*) and tis suppos'd that the angle FGD is also equal to the angle EFB ; therefore the alternate angles AFG, FGD, will be equal ; and (*by the 27.*) the lines AB, and CD, will be parallel.

I say in the second place, that if the angles BFG, and FGD, which are the internal angles on the same side, be equal to two right ones, the
lines

lines AB and CD will be parallel.

Demonstr. The angles AFG and BFG are equal to two right angles, (*by the 13.*) and 'tis suppos'd that the angles BFG, and FGD, are also equal to two right angles; therefore the angles AFG, BFG, are equal to the angles BFG, and FGD; therefore taking away the angle BFG, which is common to both, the alternate angles AFG and FGD will be equal; and (*by the 27.*) the lines AB and CD will be parallel.

PROPOSITION XXIX.

A THEOREM.

If a line cut two parallels, the alternate angles will be equal; the external angle will be equal to the internal opposite angle; and the two internals on the same side will be equal to two right angles..

LET the line EH [*see fig, preced.*] cut the two parallels AB, and CD; I say first, the alternate angles AFG, and FGD, are equal. From the points F and G draw the perpendiculars GA, and FD, which by the definition of Parallels are equal.

Demonstration.

In the rectangle triangles AFG, and FGD,
the

the sides FD and AG being equal, as also the right angles A and D , and the side FG common to both: I say first, that the side GD is equal to AF . For if GD be greater; having cut the line DI equal to AF , and drawn the line FI ; the triangles AFG , and FDI , would have their bases GF and FI equal, which is impossible. For since the angle D is a right angle, the angle FID is an acute, and FIG an obtuse, (*by the 13.*) therefore (*by the 18.*) in the triangle FIG , the side FG oppos'd to the obtuse angle, is greater than FI . Therefore DG is equal to AF ; and the triangles AFG and FGD , having all their sides equal, will have the alternate angles AFG and FGD equal, as being oppos'd to the equal sides AG , and FD .

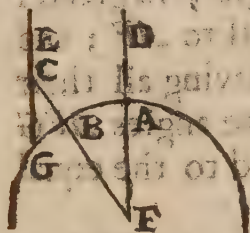
I say again, that the external angle EFB is equal to the internal EGD , because (*by the 15.*) it is equal to its opposite AFG , which is equal to its alternate FGD .

Lastly, since the angles AFG and GFB are equal to two right ones; taking away AFG , and substituting in its place its alternate FGD , the two internal angles GFB , and FGD , will be equal to two right angles.

The Use.

‘Eratostenes found out by these Propositions a way of measuring the circuit or circumference

'rence of the Earth. In order to which he sup-
 'pos'd two rays, proceeding from the center co-
 'the Sun to two points of the earth, to be phy-
 'sically parallel; and also that at *Syene*, a town
 'in the higher parts of *Egypt*, the Sun comes
 'exactly to the Zenith upon the day of the Sol-
 'stice, observing the Wells there to be there-
 'illuminated to the very bottom: and likewise
 'computed the distance between *Alexandria*
 'and *Syene* by miles, or furlongs.



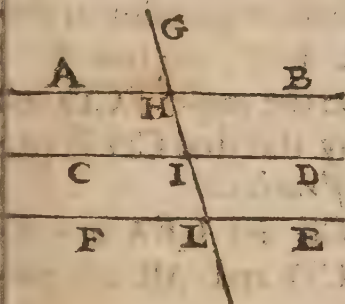
'Let us therefore suppose *Sy-
 'ene* to be at the point *A*, and
 '*Alexandria* at *B*, where we
 'erect a style *BC* perpendicu-
 'lar to the Horizon; and let
 'the two lines *DF* and *EG* re-
 'present the two rays proceeding from the cen-
 'ter of the Sun upon the day of the Solstice
 'which are parallel to each other. *DA*
 'which passes by *Syene*, is perpendicular, that
 'is, it passes through the center of the earth.
 'Having observ'd by the perpendicular style
 '*BC* the angle *GCB*, made by the ray of the
 'Sun *EG*; I say, the rays *DA* and *EG* being paral-
 'lel, the alternate angles *GCB* and *BFA* are
 'equal; by which means we have got the angle
 '*AFB*, and its measure *AB*; which gives us in
 'degrees the distance between *Alexandria* and
 '*Syene*. And having suppos'd it to be known
 'in miles, the circumference of the earth may

be found by the simple Rule of Three, saying,
If so many degrees give so many miles, how many
will 360 give?

PROPOSITION XXX.

A THEOREM.

Lines parallel to a third, are also parallel among themselves.



Suppose the lines AB, and FE to be parallel to the line CD: I say, they are parallel betwixt themselves. Let the line GL cut them all three.

Demonstration.

For as much as the lines AB and CD are parallel, the alternate angles AHI, and HID, are equal (*by the 29.*) and because the lines CD and FE are also parallel, the external angle HID will be equal to the internal ILE; [*by the same*] therefore the alternate angles AHI, and ILE, will be equal, and the lines AB and FE parallel [*by the 27.*]

I 2 PROP.

PROPOSITION XXXI.

A PROBLEM.

To draw a line parallel to another by a point given

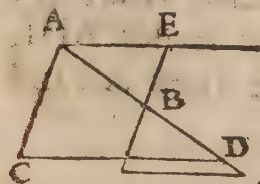


LET it be requir'd to draw a line by the point C which shall be parallel to the line AB. Draw the line CE and make the angle ECD equal to the angle CEA. I say the line CD is parallel to AB.

Demonstration.

The alternate angles DCE and CEA are equal: therefore the lines CD and AB are parallel.

'The eleventh Maxime, i. e. If a line falling upon two others makes the internal angles less than two right angles, those lines will concur, may also now be easily demonstrated.



Let the line AC, falling upon the lines AB and CD, make the internal angles ACD, and CAB, less than two right angles: I say that the lines AB and CD will concur. Let the angles AGD and CAE be equal to two right angles: the lines AE and CD will be parallel.

[6]

[by the 28.] Take the line AB as long as you please, and by the point B draw EF parallel to CA. Then take the line EB so oft as is necessary, to make it reach lower than the line CD; as in the present figure I have taken it only twice; so that EB and BF are equal. By the point F draw a parallel FG equal to AE, and joyn the line GB. I say that the line ABG is only one line; and that therefore the line AB concurring in FG, if the line CD be continu'd, since it cannot cut its parallel FG, it will cut the line BG between B and G.

Demonstration.

The triangles AEB and BFG have the sides AE and FG, BE and BF, equal; as also the alternate angles AEB, and BFG, (by the 29.) therefore they are equal in all respects, (by the 4.) And the opposite angles ABE, and FBG, are equal; and by consequence [by the coroll. of the 15.] AB and BG make but one right line.

The Use.

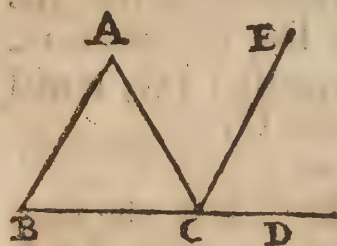
'The use of parallel lines is very common; as in *Perspectives*, for as much as the appearances or images of lines parallel to the picture or table, are parallel among themselves. In *Navigation*, the lines of the same Rhomb of the wind are describ'd by Parallels. *Polar Dials* have the hour-lines parallels. The Compass of Proportion is founded also upon parallels.

PROP

PROPOSITION XXXII.

A THEOREM.

The external angle of a triangle is equal to both the internal opposite angles, taken together; and all the three angles of a triangle are equal to two right angles.



LET the side BC of the triangle ABC be produc'd to D: I say, that the external angle ACD is equal to both the internal angles A and B taken together. By the point C draw the line CE parallel to the line AB.

Demonstration.

The lines AB and CE are parallels, therefore [by the 29.] the alternate angles ECA and CAB are equal; and [by the same] the external angle ECD is equal to the internal B. And by consequence the whole angle ACD, being equal to both the angles ACE, and ECD, of which it is compos'd, will be equal to both the angles A and B taken together.

In the second place. The angles ACD and ACB are equal to two right angles, (by the 13.) and I have demonstrated the angle ACD to be equal to both the angles A and B taken together;

ther; therefore the angles ACB, A, and B, that is to say, all the angles of the triangle ABC, are equal to two right angles, or, which is all one, to 180 degrees.

Corollary 1. All the three angles of one triangle are equal to all the three angles of another triangle,

Coroll. 2. If two angles of one triangle be equal to two angles of another triangle, their third angles are also equal.

Coroll. 3. If a triangle has one right angle, the other two will be acute; and taken together will be equal to one right angle.

Coroll. 4. From a point given onely one perpendicular can be drawn to the same line; because a triangle cannot have two right angles.

Coroll. 5. A perpendicular is the shortest of all the lines, that can be drawn from the same point to the same line.

Coroll. 6. In a rectangle triangle the right angle is the greatest angle, and the side oppos'd to it the greatest side.

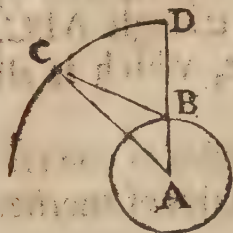
Coroll. 7. Every angle of an equilateral triangle contains 60 degrees; that is to say, the third part of 180.

The U S E.

'This Proposition is of use in *Astronomy* to determine the *Parallax*. Suppose the point

A

' A to be the center of the earth ; and that from
 ' the point B upon the superfi-
 ' cies be taken the angle DBC,,
 ' that is to say, the distance of a
 ' star from the Zenith D. If the
 ' earth was transparent the star
 ' would appear remote from the



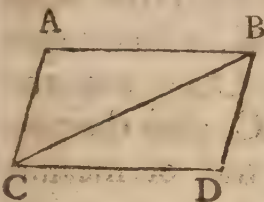
' Zenith D, according to the bigness of the an-
 ' gle CAD, which is less than the angle CBD..
 ' For the angle CBD being an external angle:
 ' in respect of the triangle ABC, it is (by the
 ' 32.) equal to both the opposite angles A and
 ' C. Therefore the angle C will be equal to
 ' the excess of the angle CBD above the angle
 ' A. Whence I infer, that if I can know by
 ' the *Astronomical tables* how far remote from
 ' the Zenith the star ought to appear to him
 ' that should be at the center of the earth, and
 ' observe it at the same time from the superficies,,
 ' the difference of those two angles will be the
 ' Parallax BCA.

PROP.

PROPOSITION XXXIII.

A THEOREM.

Two lines drawn towards the same parts, from the extremity of two other lines that are equal and parallel, are also themselves equal and parallel.



LET the lines AB and CD be parallel and equal; and let the lines AC and BD be drawn from their extremities towards the same parts;

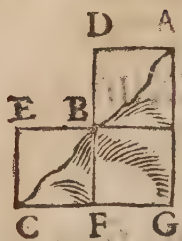
I say, that the lines AC and BD are equal and parallel. Draw the Diagonal line BC.

Demonstration.

Since the lines AB and CD are parallel, the alternate angles ABC and BCD will be equal (*by the 29.*) therefore in the triangles ABC and BCD, which have the side BC common, and the sides AB and CD equal, together with the angles ABC and BCD equal also, the bases AC and BD will be equal, (*by the 4.*) and also the angles DBC, and BCA; which being alternate angles, the lines AC and BD will be parallel, (*by the 27.*)

The USE.

This Proposition is reduc'd to practice for
K the



the measuring the perpendicular heights, AG , of the valliest Mountains; and also their horizontal lines, CG , which are hidden by their bulk. Take a large square $ADBF$, and place it so at the point A , that the side DB may fall perpendicularly; then measure the sides AD and DB . This done, do the same again at the point B , and measure BE and EC : the sides parallel to the horizon, AD , BE , added together give the horizontal line CG ; and the perpendicular sides DB and EC , give the perpendicular height AG . This way of measuring is called * *Cultellatibn*.

* *Measuring by piece-meal.*

PROPOSITION XXXIV.

A THEOREM.

The opposite sides and angles of a Parallelogram are equal; and the diameter divides it into two equal parts.

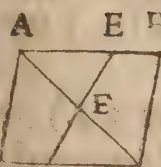
SUPPOSE the figure $ABDC$ [see the fig. of the preceding Prop.] to be a Parallelogram, that is to say, that the sides AB , CD ; AC , and BD , are parallel. I say, the opposite sides AB , CD ; AC , and BD , are equal; as also that the angles

A and D, ABD, and ACD; and that the diameter BC equally divides the whole figure.

Demonstration.

The lines AB, and CD, are suppos'd to be parallels: therefore the alternate angles ABC and BCD will be equal, (*by the 29.*) In like manner the sides AC and BD being suppos'd to be parallels, the alternate angles ACB and CBD will be equal. And further, the triangles ABC, BCD, having the same side BC; and the angles ABC, BCD; ACB, and CBD equal, will be equal in all respects, (*by the 26.*) Therefore the sides AB, CD; AC, and BD, and the angles A and D, are equal: and the diameter BC divides the figure into two equal parts. And since the angles ABC, BCD; ACB, and CBD, are equal, joyning together ABC and CBD; and likewise BCD and ACB, we infer that the opposite angles ABD, and ACD are equal.

The Use.



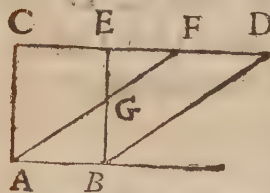
Surveyors have need of this Proposition for dividing grounds. If the field be a Parallelogram, they can divide it into two equal parts by the diameter AD. But if you be oblig'd to divide it by the point E: divide first the diameter into two equal parts by the point F, then draw the line EFG, which will divide the figure into two equal parts. For

the triangles AEF and GFD , having the alternate angles EAF, FDG ; and AEF, FGD ; and the sides AF and FD equal, are equal, (*by the 26.*) And since the Trapezium $BEFD$ with the triangle AEF ; that is to say, the triangle ADB , is half the parallelogram, (*by the 34.*) the same trapezium $BEFD$ with the triangle DGF will be half the same. Therefore the line EG divides it in the middle.

PROPOSITION XXXV.

A THEOREM.

Parallelograms, having the same base, and being between the same parallels, are equal.



LET the Parallelograms be $ABEC$, and $ABDF$, having the same base AB , and being between the same parallels AB and CD . I say, they are equal.

Demonstration.

The sides AB, CE , are equal, (*by the 34.*) as also AB, FD : therefore CE and FD are equal; and adding to them EF , the lines CF and ED will be equal. The triangles therefore CFA and EDB , have the sides CA, EB , as also CF , and ED equal, together with the angles DEB , and FCA , (*by the 29.*) one being an external, and the

the other an internal angle on the same side. Therefore (*by the 4.*) the triangle ACF and BED are equal; and taking from them both, that which is common, *viz.* the little triangle EGF , the trapezium $FGBD$ will be equal to the trapezium $CAGE$: and adding to both the triangle AGB , the Parallelograms $ABEC$ and $ABDF$ will be equal.

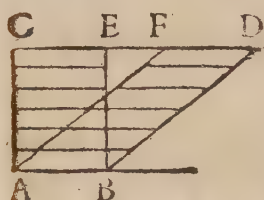
The use.

‘*Scotus*, and some Divines since him, have made use of this Proposition to prove, that *Angels may extend themselves to what space they please.* For supposing they can assume any figure, provided they have not a greater extension: it is evident, that if an Angel should possess the space of the Parallelogram $ABEC$, it may likewise occupy the space of the Parallelogram $ABDF$; and because parallels may be continu’d *in infinitum*, (without end,) and Parallelograms may be still form’d longer and longer, which will all be equal to $ABEC$; an Angel will be able to extend it self still farther and farther.

A Demonstration of the same Proposition by Indivisibles.

‘This method was lately invenred by *Cavalierius*; which has found different acception in the

the world, some approving, and others reject-
 ing it. His method consists in this ; that we
 imagine superficies's to be compos'd of lines,,
 like so many threds. And tis certain, that
 two pieces of linnen will be equal, if they have
 both the same number of threds, of equall
 length, and equally compacted.



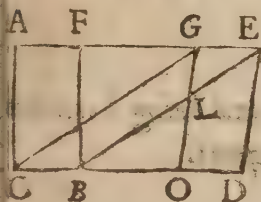
Let two Parallelogramss
 therefore ABEC, and ABDF,,
 be propos'd, having the same
 base AB, and being between the
 same parallels AB, CD. Dividee
 the parallelogram ABEC into
 as many lines as you please, parallel to AB,,
 which continue to the other parallelogram
 ABDF. Tis evident there will be no more in
 one, than in the other ; and that they will be
 of equal length, being all equal to the base
 AB ; and that they will not be more closely
 compacted in one, than in the other : there-
 fore the parallelograms will be equal.

PROP.

PROPOSITION XXXVI.

A THEOREM.

Parallelograms, upon equal bases, and between the same parallels, are equal.



LET the bases CB and OD of the parallelograms ACBF, ODEG, be equal; and let both be between the same parallels AE, CD. I say the parallelograms are equal. Draw the lines CG, and BE.

Demonstration.

The bases CB, and OD, are equal: OD, and GE, are also equal: therefore CB and GE are equal, and parallel; and by consequence (according to the 33.) CG and BE will be equal and parallel; and CBEG will be a parallelogram equal to CBFA, (by the 35.) having both the same base. In like manner, taking GE for the base, the parallelograms CODE and CBEG will be equal, (by the same.) Therefore the parallelograms ACBF, and ODEG, are equal.

The use.

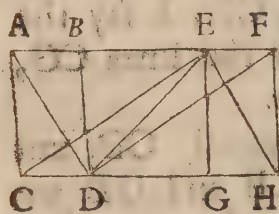
‘ We oft reduce parallelograms, which have oblique angles, as CBEG, or ODEG, to rectangles; as CBFA: so that measuring the latter,

‘latter, which is easie, being only to multiply
 ‘AC by CB, the product being equal to the
 ‘parallelogram ACBF, we may by consequence
 ‘know the other parallelograms CBEG, or
 ‘ODEG.

PROPOSITION XXXVII.

A THEOREM.

Triangles having the same base, and being between the same parallels, are equal.



IF the triangles ACD and CDE, have the same base CD, and be inclos'd between the same parallels AF, and CH, they will be equal. Draw the lines DB, and DF, parallel to the lines AC, and CE, and you will have form'd two parallelograms.

Demonstration.

The Parallelograms ACDB, and ECDF, are equal (*by the 35.*) and the triangles ACD, CDE, are the halves of those Parallelograms, (*by the 34.*) Therefore the triangles ACD, CDE, are equal. \square

PROP.

PROPOSITION XXXVIII.

A THEOREM.

Triangles, that have equal bases, and are inclos'd within the same parallels, are equal.

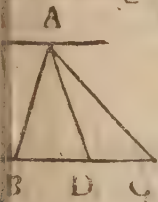
IF the triangles ACD, and ECH, [see fig. *preced.*] have equal bases CD, and GH, and are inclos'd within the same parallels AF, and EH, they are equal. Draw the lines BD and HF parallel to the sides AC, and EG; and you will have form'd two parallelograms.

Demonstration.

The Parallelograms ACDB, and EG'HF, are equal, (*by the 36.*) and the triangles ACD and EGH are the halves of those parallelograms, (*by the 34.*) therefore they are also equal.

The U S E.

'We have in these propositions directions for dividing a triangular field into two equal parts;



'for example the triangle ABC.

'Divide the line which you will

'take for the base, as BC, into two

'equal parts in D: I say the trian-

'gles ABD, and ADC, are equal.

For if you suppose a line drawn by A, parallel to BC, those triangles will have equal bases,

L

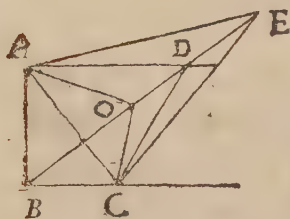
'and

'and be inclos'd within the same parallels, and
'by consequence will be equal. Other Divisi-
'ons, grounded upon the same propositions,
'might be made; but I omit them, that II
'might not be tedious.

PROPOSITION XXXIX.

A THEOREM.

*Equal triangles, upon the same base, are within
the same parallels.*



IF the triangles ABC, and
DBC, having the same
base BC, be equal; the line
AD drawn by the tops will
be parallel to the base. For
if AD and BC be no paral-
lel; if you draw a parallel by the point A, it
will fall either below the line AD, as AO; or
above it, as AE. Suppose it to fall above, and
produce BD till it meet the line AE, at the
point E; then draw the line CE.

Demonstration.

The triangles ABC and EBC are equal, (by
the 38.) since the lines AE and BC are paral-
lel; 'tis likewise suppos'd that the triangles
ABC, and DBC, are equal: therefore the tri-
angles

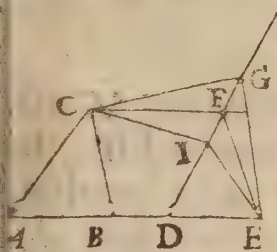
angles DBC and EBC would be equal ; which is impossible, the first being part of the second. Whence I conclude, that a line parallel to BC cannot be drawn above AD, as AE.

I add, that that parallel cannot be below AD, as AO : because the triangle BOC would be equal to the triangle ABC, and by consequence to the triangle DBC ; that is to say, the part would be equal to the whole. It must therefore be confess'd, that the line AD is parallel to the line BG.

PROPOSITION XL.

A THEOREM.

Equal triangles, having equal bases, if they be taken upon the same line, are between the same parallels.



IF the equal triangles ABC and DEF, have equal bases AB and DE, taken upon the same line AE ; the line CF drawn by their tops will be parallel to AE. For if it be not parallel, having drawn by the point C a line parallel to AE, it will pass either above CF, as CG ; or below it, as

CI. *Demonstration.*

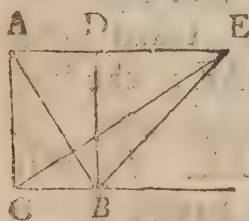
If it pass above CF, as CG, continue DF till

it meet with CG in G ; and draw the line EC . The triangles ABC and DGE would be equal (by the 38.) and ABC and DEF being supposed to be equal, DEF , DGE , would be also equal, which, one being part of the other, cannot possibly be: therefore the parallel cannot pass above CF . I add, that neither can it pass below it, as CI ; because then the triangles ABG and DEI would be equal, and by consequence DEI and DEF , the part and the whole. Therefore only CF can be parallel to AE .

PROPOSITION XLI.

A THEOREM

A parallelogram will be double to a triangle, if they be between the same parallels, and have equal bases.



IF the parallelogram $ACBD$ and the triangle EBC , be between the same parallels AE and BC ; and have the same base BC , or only equal bases; the parallelogram will be double the triangle. Draw the line AC .

Demonstration.

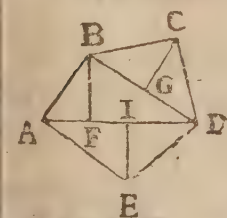
The triangles ABC and BCE are equal, (by the 38.)

the 37.) But the parallelogram $ACFD$ is double the triangle ABC , (by the 34.) It is therefore double the triangle BCE . It would be also double a triangle, that, having a base equal to BC , should be between the same parallels.

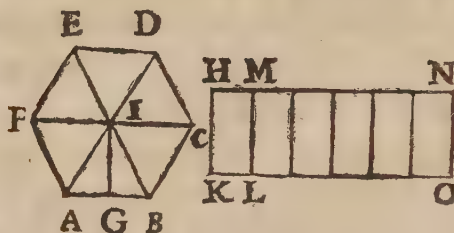
The USE.

H A G 'The ordinary method of measuring the *area* or superficies of a triangle is built upon this proposition. If the triangle ABC be propos'd; from the angle A we must draw AD perpendicular to the base BC ; then multiplying the perpendicular AD by half the base BE , the product gives the *area* of the triangle: because multiplying AD , or, what is the same, EF by BE , we have a rectangle $BEFH$, which is equal to the triangle ABC . For (by the 41.) the triangle ABC is half the rectangle $HBCG$; and so likewise is the rectangle $CEFH$.

'We measure all sorts of rectilineal figures, as $ABCDE$ by dividing them, into triangles. as BCD , ABD , AED ; drawing the lines AD , and BD ; and the perpendiculars CG , BF , and EI . For multiplying half of BD by CG , and half of AD by BF , and by EI , we have the *area* of all those triangles: adding which together the summe is equal to the rectilineal figure $ABCDE$. We



We find the *area* of regular Polygons, by multiplying half their circuit by a perpendicular drawn



from their center to one of their sides. For multiplying AI by IG , we

shall have a rectangle $HKLM$ equal to the triangle AIB : and repeating the same for all the other triangles, taking always half of the bases, we shall have a rectangle $HKON$, which will have the side KO compounded of all the half bases, and by consequence equal to half the circumference; and the side HK equal to the perpendicular IG .

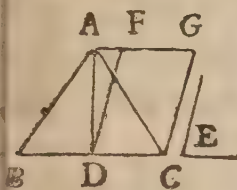
'Tis according to this principle, that *Archimedes* has demonstrated, that a circle is equal to a rectangle compris'd under the semidiameter, and a line equal to half the circumference.

PROP.

PROPOSITION XLII.

A PROBLEM.

To make a Parallelogram equal to a triangle given, having one angle equal to an angle given.



LET a parallelogram be described, equal to the triangle ABC, and having an angle equal to the angle E, divide the base BC into two equal parts at the point D; and draw the line AG parallel to BC, (by the 31.) then make the angle GDF equal to E, (by the 23.) and lastly, draw the line CG parallel to DF: the figure FDCG is a parallelogram, because the lines FG, DC; FD, and GC, are parallels, and its angle CDF is equal to the angle E; and further, 'tis also equal to the triangle ABC.

Demonstration.

The triangle ADC is half the parallelogram FDCG, (by the 41.) 'tis also half the triangle ABC; since the triangles ADC, and ADB, are equal, (by the 38.) Therefore the triangle ABC is equal to the parallelogram FDCG.

PROP.

PROPOSITION XLIII.

A THEOREM.

The complements of a parallelogram are equal



IN the parallelogram $ABCD$, the complements AHC , EHI , and EGI , are equal.

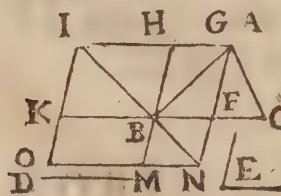
Demonstration.

The triangles AHC , and BCI , are equal, (*by the 34.*) therefore if the triangles HBE , and BIE ; FEC , and CGE , which are also equal, (*by the same,*) be subtracted, the complements AHC , $GDIE$, which remain, will be equal.

PROPOSITION XLIV.

A PROBLEM.

To describe a parallelogram upon a line given, which shall be equal to a triangle, and have such a certain angle; i.e. equal to one given.



SUPPOSE you be required to make a parallelogram, which shall have one of its angles equal to the angle E , and one of its sides equal to the

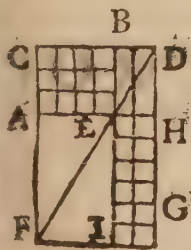
the line D, and be equal to the triangle ABC. Make the Parallelogram BFGH, (*by the 42.*) which has the angle BFG equal to the angle E, and is equal to the triangle ABC. Produce the sides GF, and GH, so that HI may be equal to the line D; and draw the line IBN till it cuts GF produc'd to N; and from the point N draw the line NO parallel to GI, and IO parallel to BH; producing also the side FB to K, and HB to M. The parallelogram MK is that which you desire.

Demonstration.

GF and HM being parallels, the alternate angles GFB or the angle E, and FBM, are equal (*by the 29.*) In like manner the lines KB and MN being parallel, the alternate angles FBM, and BMO, are equal; therefore the angle BMO is equal to E, and the side KB is equal to the line HI, or D: and lastly, the parallelogram MK is equal to the parallelogram BFGH, (*by the preceding,*) and that was made equal to the triangle ABC. Therefore the parallelogram MK is equal to the triangle ABC.

The USE.

This Proposition contains a kind of Geometrical Division: for in Arithmetical Division a number is propos'd, which may be lookt on as a rectangle: for example the rectangle AB, consisting of
M twelve



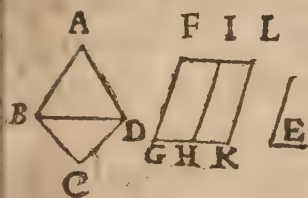
' twelve square feet, which is to be divided by
 ' another number, suppose, two ; that is to say,
 ' another rectangle is desir'd to be made equal
 ' to AB, having one of its sides, BD, equal to
 ' two ; and the question is, how many feet the
 ' other side ought to contain; which is, as it were,
 ' the quotient. This is done Geometrically
 ' by the Rule and Compass. Take BD consist-
 ' ing of two feet, and draw the Diagonal DEF:
 ' the line AF is that which you seek. For, hav-
 ' ing completed the rectangle DCFG, the com-
 ' plements EG, and EC, are equal, (*by the 43.*)
 ' and EG has for one of its sides EH, equal to
 ' BD, of two feet in length ; and EI equal to
 ' AF. This kind of Division is call'd *Applica-*
 ' *tion*, because the rectangle AB is apply'd to
 ' the line BD, or EH : and from hence tis, that
 ' all Division is frequently call'd *Application* ;
 ' because the ancient *Geometricians* made more
 ' use of the Rule and Compass, than of *Arith-*
 ' *metick*.

PROP.

PROPOSITION XLV.

A PROBLEM.

To describe a parallelogram, which shall have a certain angle; and be equal to a rectilineal figure given.



LET the rectilineal figure propos'd be $ABCD$, to which you are required to make an equal parallelogram, which shall

have an angle equal to the angle E . Divide the rectilineal into triangles by the line BD : and (by the 42.) make a parallelogram $FGHI$, which has the angle IHG equal to the angle E , and is equal to the triangle ABD ; and (by the 44.) make the parallelogram $IHKL$ equal to the triangle BCD , having one side equal to IH , and the angle LIH equal to the angle E . The parallelogram $FGKL$ will be equal to the rectilineal $ABCD$.

Demonstration.

Nothing need be prov'd, but that the parallelograms $FGHI$, and $HKLI$, make up but one; that is to say, GH , and HK , make but one right line. The angles GHI , and LKH , are equal to the angle E . And the angles LKH ,

M 2

and

and KHI, are equal to two right angles, because KHIL is a parallelogram. Therefore the angles GHI and KHI are equal to two right angles and (by the 14.) GH and HK make one right line.

The Use.

'The use of this proposition is the same with the preceding ; serving to measure the capacity of any figure whatsoever , by reducing it into triangles, and then making a rectangular parallelogram equal to them.

'Tis easie likewise to make a rectangular parallelogram upon a determinate side, which may be equal to many irregular figures. In like manner having many figures a Rectangular may be described equal to their difference.

PROPOSITION XLVI.

A PROBLEM.

To describe a square upon a line given.



TO describe a square upon the line AB, draw two perpendiculars AC and BD equal to AB and draw the line CD.

Demonstration.

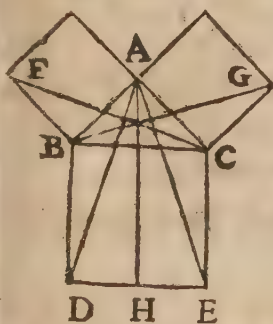
The angles A and B being right angles

angles, the lines AC and BD are parallels (by the 28.) They are also equal; therefore the lines AB and CD are parallels and equals, [by the 33.] and the angles A and C equal to two right angles; as also B and D, [by the 29.] and since A and B are both right angles, the angles C and D will be so likewise. Therefore the figure AD has all its sides equal, and all its angles right angles, and by consequence is a square.

PROPOSITION XLVII.

A THEOREM.

The square of the base of a rectangular triangle, is equal to the squares of both the other sides taken together.



Suppose the angle BAC to be a right angle, and that squares were described upon all the sides BC, AB, and AC: that of the base BC, which is oppos'd to the right angle, will be equal to the squares of both the sides AB, and AC. Draw the line AH parallel to BD, and CE; and joyn the lines AD, AE, FC, and BG. I will prove the square AF is equal to the rectangle BH, and the square AG to the rectangle

gle CH; and therefore the square BE is equal to both the squares AF and AG.

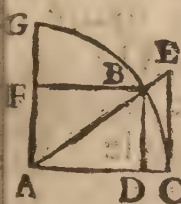
Demonstration.

The triangles FBC, and ABD, have the sides AB, BF; BD, and BC, equal: and the angles FBC, and ABD, are equal, each containing the angle ABC more than their respective right angles. Therefore [by the 4.] the triangles ABD and FBC are equal. But the square AF is double the triangle FBC, [by the 41.] having the same base BF, and being between the same parallels BF, and AC. In like manner the rectangle BH is double the triangle ABD, having likewise the same base BD, and being between the same parallels BD and AH. Therefore the square AF is equal to the rectangle BH. By the same method the triangles ACE, and GCB, may be prov'd to be equal, [by the 4.] and the square AG to be double the triangle GCB; and the rectangle CH, double the triangle ACE, [by the 41.] therefore the square AG is equal to the rectangle CH; and by consequence the squares AF and AG are equal to the square BDEC.

The U S E.

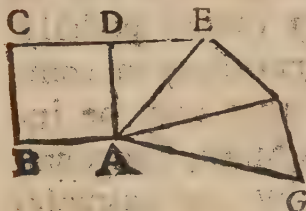
'Tis said that *Pythagoras*, having found out this Proposition, sacrific'd a *Hecatomb*, i. e. a hundred Oxen, to the *Muses*, to return them thanks for their assistance; supposing it, it seems

seems, above the power of bare humane invention. Nor was his esteem thereof so irrational, as to some perhaps it may appear; this proposition being the foundation of a very considerable part of the *Mathematicks*. For in the first place *Trigonometry* cannot possibly subsist without it, it being necessary to compose a table of all the lines that may be inscrib'd in a circle, as Chords, Sines, Tangents, Secants; as may appear by one example.



Suppose the semidiameter AC to be divided into 100000 parts, & that the arch BC contains 30 degrees. Since the chord, or line that subtends 60 degrees is equal to the semidiameter AC; BD the sine of 30 degrees, will be equal to half AC, and therefore contain 50000 parts. Now in the rectangular triangle ADB, the square of AB is equal to the squares of AD, and BD. Make therefore the square of AB, by multiplying 100000 by 100000, and from the product subtract the square of 50000 or BD; the remainder will be the square of AD, or BF the sine of the complement: and extracting the square root of that number, you will have the line FB. This done, making as AD to BD, so AC to CE, you will have the tangent CE; then adding together the squares of AC and CE, the product (*by the 47.*) will give the

the square of AE; extracting therefore from
that number the square root, you will know
the length of the line AE, which is the se-
cant.



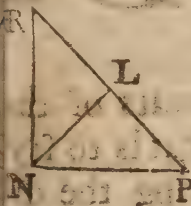
By this also we may aug-
ment figures, as much as
we please. For example;
to double the square ABCD,
D, continue the side CD,
so that AD and DE may
be equal: the square of
AE will be double the square ABCD; since
(by the 47.) it is equal to both the squares of
AD and DE. Making the right angle AEF,
and taking EF equal to AB, the square of AE
will be triple the square ABCD. Again
making the right angle AFG, and taking FG
equal to AB, the square of AG will be quad-
ruple, or four times the square of ABCD.
And that which I say of the square, may be
understood of all similar figures.

PROP

PROPOSITION XLVIII.

A THEOREM.

If in a triangle the square of one side be equal to the squares of both the other sides, taken together; the angle opposite to that first side will be a right angle.



IF the square of the side NP be equal to both the squares of the sides NL, and LP, taken together; the angle NLP will be a right angle. Draw LR perpendicular to NL, and equal to LP; then draw the line NR.

Demonstration.

In the rectangular triangle NLR, the square of NR is equal to the squares NL, and RL, or LP, (*by the 47.*) Now the square of NP is also equal to the same squares of NL and LP; therefore the square of NR is equal to the square of NP, and by consequence the lines NR and NP are equal. And because the triangles NLR, and NLP, have the side NL common; the sides LP and LR equal, and their bases NP and NR also equal; the angles NLP and NLR will be equal, (*by the 8.*) and the angle NLR being a right angle, the angle NLP must be so too.

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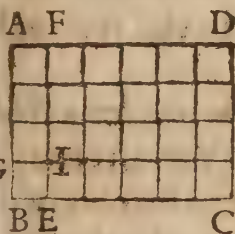
THE SECOND BOOK
OF THE
ELEMENTS
OF
EUCLID.

' *EUCLID* in this Book treats of the
 ' powers of right Lines; that is to say,
 ' of their Squares; comparing the di-
 ' vers Rectangles, which are made upon a Line
 ' divided, as well with the Square, as the
 ' Rectangle, of the whole Line. 'Tis a part
 ' exceeding useful, serving for the foundation
 ' of the principal Operations of *Algebra*. The
 ' three first Propositions demonstrate the third
 ' Rule, or Operation of *Arithmetick*, *Multi-*
 ' *plication*. The fourth teaches to extract the
 ' square Root of any number whatsoever. Those
 ' that follow to the Eight serve upon many oc-
 ' casions in *Algebra*. The rest instruct us in O-
 ' perations proper for *Trigonometry*. This Book
 ' seems at first view very difficult; because
 ' men are apt to imagine there is something my-
 ' sterious contain'd therein; nevertheless the
 ' great

greatest part of its Demonstrations are grounded on this most evident Principle, that the whole is equal to all its parts taken together. But it ought not to discourage any, if they should not at the first attempt fully comprehend them.

DEFINITIONS.

A rectangular parallelogram is compris'd under two lines, that form a right angle.

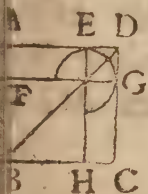


Observe that henceforward by a rectangle we shall intend such a parallelogram, whose angles are all right angles, distinguishing it by giving its longitude and latitude, naming two of its sides, which contain one of its angles, as the lines AB and BC. For the rectangle ABCD is compris'd under the lines AB and BC; BC denoting its longitude and AB its latitude; and the other being equal to these it will not be necessary to name them. I have also formerly intimated that the line AB, remaining perpendicular to BC, and being mov'd from one extremity thereof to the other, produces the rectangle ABCD; and that that

' motion has some resemblance to *Arithmetical*
 ' multiplication : so that, as the line AB mov-
 ' ing over the line BC, that is, taken so many
 ' times, as there are points in BC, composes
 ' the rectangle ABCD : so the multiplication
 ' also of AB by BC, will give the rectangle
 ' ABCD. As, suppose I knew the number of
 ' *Mathematical points*, that are in AB, for
 ' example 40, and that there were 60 in BC :
 ' it is evident that the rectangle ABCD will
 ' have so many lines equal to AB, as there are
 ' points in BC; and that multiplying 40 by
 ' 60, the product will be 2400, which is the
 ' number of *Mathematical points* in the rectangle
 ' ABCD.

' I may take what quantity I please for a *Ma-*
 ' *thematical point* ; provided I do not afterwards
 ' subdivide it ; it must therefore be observ'd ;
 ' that when I measure a line, for a *Mathemati-*
 ' *cal point* I take that measure which best suits
 ' with my occasions ; for example, when I say
 ' a line of five foot in length, my *Mathema-*
 ' *tical point* is a line of a foot long ; which I
 ' take without considering that it is compos'd
 ' of any parts. In measuring a superficies like-
 ' wise I do the same, taking some known super-
 ' ficies, for example, a foot square ; which I do
 ' not afterwards subdivide. I make use of a
 ' square rather than any other figure ; because
 ' its length and breadth being equal there is no
 ' need

need of naming more than one of its dimensions to describe it. Accordingly when I would mark out the *Area* of the rectangle ABCD, I do not consider the sides as simple lines, but as rectangles of a determinate breadth: for example, when I say that the rectangle ABCD has the side AB of four foot long, since a foot is to me instead of a *Mathematical point*, I conceive the side AB to have also a foot in breadth, and to be as the rectangle ABEF. Therefore knowing how many times the breadth BE is contain'd in the line BC, I shall know how many times the line AB is contain'd in the rectangle ABCD; that is to say, multiplying AB which has four foot square by 6, the product will be 24 foot square. In like manner knowing the magnitude of the rectangle ABCD to be 24 foot square, and one of its sides AB to be 4; dividing 24 by 4 the quotient will give me the other side BC, consisting of six foot square.



2. Having drawn the diameter of a rectangle, one of the lesser rectangles thro which it passes, together with the two complements, is call'd the Gnomon. 'As the rectangle EG, 'thro which the diameter BD passes, together with the complements EF and GH, is call'd the Gnomon; their figure together representing a Carpenters square.

P R O P.

PROPOSITION I.

A THEOREM.

If two lines be propos'd, whereof one is divided into divers parts, the rectangle contain'd under those two lines is equal to the rectangles contain'd under the line which is not divided, and the parts of the line divided.

C G H D **L** ET the lines propos'd be **AB**, and **AC**; and let **AB** be divided into as many parts as you please. The rectangle **AD** contain'd under the lines **AB** and **AC**, is equal to the rectangle **AG** contain'd under **AC** and **AE**; to the rectangle **EH** contain'd under **EG** equal to **AC**, and **EF**; and to the rectangle **FD** contain'd under **FH** equal to **AC** and **FB**

*Demonstration.*

The rectangle **AD** is equal to all its parts taken together; which are the rectangles **AG**, **EH**, and **FD**; and no other. Therefore the rectangle **AD** is equal to the rectangles **AG**, **EH** and **FD** taken together.

By Numbers.

This proposition holds true likewise in numbers. Suppose the line **AC** to be five foot long; **AE** two, **EF** four, **FB** three; and by consequence **AB** nine: the rectangle contain'd under

under AC five; and AB nine, that is to say, five times nine, which makes forty five, is equal to twice five or ten, four times five or twenty, and three times five, or fifteen; for ten, twenty, and fifteen, make forty five.

The USE.

A,	53
B,	8
C,	50,3
D,	24
E,	400
F,	424

By this proposition is demonstrated the ordinary operation of multiplication. For example, if you were to multiply the number A, which is 53, by the number B, that is 8. Divide the number A into so many parts as there are characters: that is, two, 50, and 3; which multiply by 8, saying, eight times three is twenty four; and so you make one rectangle. Then multiplying the number 50 by 8, the product will be 400. But 'tis evident that the product of eight times 53, being 424, is equal to the product of 24, and the product 400 taken together.

PROP.

PROPOSITION II.

A THEOREM.

The square of any line is equal to the rectangle contain'd under the whole line, and all its parts.

C G H D



A E F B

LET the line propos'd be AB and its square ABDC. I say the square ABDC is equal to the rectangle contain'd under the whole line AB, and AE; another under AB and EF; and a third under AB and FB.

Demonstration.

The square ABDC is equal to all its parts taken together, which are the rectangles AG, EH, FD. The first AG is contain'd under AC equal to AB, and AE. The second EH is contain'd under EG equal to AC or AB, and FE. The third FD is contain'd under FH equal to AB, and FB: and 'tis the same thing to be contain'd under a line equal to AB, and to be contain'd under AB it self. Therefore the square of AB is equal to the rectangle contain'd under AB, and AE, EF, FB the parts of AB.

By Numbers.

Let the line AB represent the number nine

square will be 81. Let also the part AE four ; EF three ; and EC two : nine times four make thirty six ; nine times three twenty seven ; and nine times two eighteen ; and 'tis plain that 36, 27, and 18 make 81.

The USE.

This proposition serves likewise to prove multiplication ; as also for Equations in *Algebra*.

PROPOSITION III.

A THEOREM.

a line be divided into two parts, the rectangle contain'd under the whole line, and one of its parts, is equal to the square of the same part, and the rectangle contain'd under both the parts.

E **F** **L**ET the line AB be divided into two parts at the point C ; and let a rectangle be made CB of the whole AB and one of its parts AC, that is to say, let AD be equal to AC ; and then if rectangle AF be completed, it will be equal to the square of AC, and the rectangle contain'd under AC CB. Draw the perpendicular CE.

O

De.

Demonstration.

The rectangle AB contain'd under AB and AC , is equal to all its parts, which are the rectangles AE , and CF . The first AE is the square of AC , the lines AE and AD being equal; and the rectangle CF is contain'd under CB , and CE equal to AE or AC . Therefore the rectangle contain'd under AB and AC is equal to the square of AC , and the rectangle contain'd under AB and CB .

By Numbers.

Let AB be 8; AC 3; and CB 5: the rectangle contain'd under AB and AC , will be three times eight, or 24: the square of AC 3, is nine; and the rectangle contain'd under AC 3, and CB 5, is three times 5 or 15. But it is evident that 15 and 9 make 24.

The Use.

A,	43
C,	40, 3
B,	3
120.	9.
129	

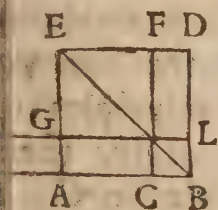
'The use of this proposition is
'to demonstrate the ordinary practice
'of multiplication. For example,
'you would multiply the number
'by 3; having divided the number
'43 into 40, and 3: three times
'will amount to as many as three times three,
'nine, that is the square of three; and three
'times 40, that is, 120; for three times four
'three is 129. Beginners ought not to be
'con

ourag'd, if they do not presently apprehend these propositions; which yet, in truth are not difficult, but as they are conceiv'd to contain some strange mystery.

PROPOSITION IV.

A THEOREM.

If a line be divided into two parts, the square of the whole line, will be equal to the squares of both the parts; and two rectangles contain'd under the same parts.



LET the line AB be divided in C, and its square ABDE describ'd; let its diagonal also EB be drawn and a perpendicular cutting it CF:

by that point let the line GL be drawn parallel to AB. 'Tis evident that the square ABDE is equal to the four rectangles GF, CG, and LF. The two first of which are the squares of AC and CB: and the two complements are contain'd under AC and

Demonstration.

The sides AE and AB are equal: therefore the angles AEB, and ABE are half right angles

angles : and because the lines GL and AB are parallels ; the angles of the triangles of the square GF (*by the 29. 1.*) will be equal ; as also their sides (*by the 6. 1.*) Therefore GF is the square of AC. In like manner CL is the square of CB : the rectangle GC is contain'd under AC, and AG equal to BL or BC ; and the rectangle LF is contain'd under LD equal to AC, and FD equal to BC.

Coroll. If you draw the diagonal of a square the rectangles which it cuts are squares.

The USE.

A,	144
B,	22
C,	12

‘ This proposition teaches the method of extracting the square root off any number propos’d. Let the number be A, or 144, represented by the square AD, and its root by the line AB. I suppose it known from other principles that it requires two characters. I imagine therefore that the line AB is divided in C, so that AC may represent the first character, and BC the second. Then searching the root of the first character of the number 144, which is 100, I find it to be 10: and making its square 100, represented by the square GF, I subtract it from 144; and there remains 44 for the rectangles GC, FL; and the square CL: But because the figure of a Gnomon is not proper for this operation, I transport the rectangle FL unto KG, making one whole rectangle.

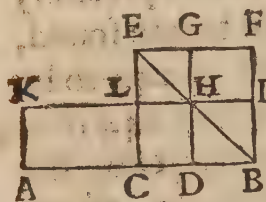
rectangle KL, that is, 44. I know also already almost the whole side KB: for AC being 10, KC must be 20: I must therefore divide 44 by 20; that is to say, for my Divisor doubling the root found; I enquire then how many times 20 I can have in 44? and find twice; and therefore take 2 for the side BL; and because 20 was not the intire side KB, but only KC; that two which came in the quotient I add to the Divisor, making it 22; which number being found precisely twice in 44, adding 2 to the root before found, I conclude the whole square root of 144 to be 12. You see then that the square 144 is equal to the square of 10, which is 100, the square of 2, that is 4; and twice 20, which make the two rectangles contain'd under two and ten.

PROP-

PROPOSITION V.

A THEOREM.

If a line be divided into two equal parts and two parts that are unequal; the rectangle contain'd under the unequal parts, together with the square of the intermediate part; is equal to the square of half the line.



IF the line AB be divided into two equal parts in C; and two unequal parts in D; the rectangle AH contain'd under the unequal segments AD, and DB, with the square CD, will be equal to the square of CB, that is, the square CF. Compleat the figure as you see; the rectangles LG and DI will be squares. (*by the coroll. of the 4.*) I will prove then that the rectangle AH, contained under AD, and DH equal to DB, with the square LG, is equal to the square CF.

Demonstration.

The rectangle AL is equal to the rectangle DF, both being contain'd under half the line AB, and DB or DH, which is equal to it. Add to both the rectangle CH; the rectangle AH will be equal to the Gnomon CBG. Add there.

therefore again to both the square LG; and the rectangle AH, with the square LG will be equal to the square CF.

By Numbers.

Let AB be 10; AC will be 5, and CB likewise; and let CD be 2, and DB 3: the rectangle contain'd under AD 7 and DB 3, that is to say 21, with the square of CD 2, that is 4, will be equal to the square CB 5, which is 25.

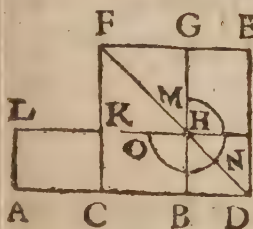
The USE.

This proposition is very useful in the third book: it is also us'd in *Algebra*, to demonstrate the manner of finding the root of an affected or impure square.

PROPOSITION VI.

A THEOREM.

If a line be divided into two equal parts, and to it another line added; the rectangle contain'd under the line compounded of those two, and that which is added, with the square of half the divided line, is equal to the square of the line compounded of that half, and the line that is added.



IF to the line AB, divided into two equal parts in C, be added the line BD; The rectangle AN, contain'd under the line AD, and DN equal to BD, with the square of

of CB, is equal to the square of CD. Make the square of CD, and having drawn the diagonal FD, draw also BG parallel to FC, cutting FD at the point H, through which passes the line HN parallel to AD. KG will be the square of CB; and BN, that of BD.

Demonstration.

The rectangles AK, and CH, being upon equal bases AC and CB, are equal (*by the 36.1.*) The complements CH and HE are equal, (*by the 43.1.*) therefore the rectangles AK and HE are equal. Add to both the rectangle CN, and the square KG: the rectangles AK and CN, that is, the rectangle AN, with the square KG, will be equal to the rectangles CN and HE, and the square KG, that is, the square CE.

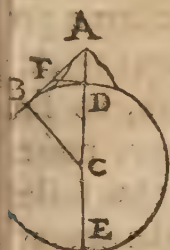
By Numbers.

Let AB consist of 8 parts; AC of 4; and CB of 4; BD of 3. so that the whole AD be 11. Tis evident the rectangle AN is three times 11, that is 33; which with the square of KG, equal to CB 4, that is 16; make 49, and therefore is equal to the square of CD 7, which is 49; for 7 times 7 make 49.

The USE.

' *Maurylorus*, by the help of this Proposition, 'measur'd the whole Earth at one single Observation.

vation. To effect which, he advises, that
 'from the top of a mountain of
 'known hight A, you observe the
 'angle BAC, made by the line
 'AB, touching the superficies of
 'the earth at B, and the line AC
 'passing through the center: and
 'that in the triangle ADF, know-
 'ing the angle A, and the right
 angle ADF, you find by *Trigonometry* the sides
 AF and FD: and because tis easie to demon-
 strate that FB and FD are equal, you will then
 know the line AB, and also its square. Now
 we have demonstrated in the preceding pro-
 position, that the line ED, being divided into
 equal parts in C, and the line AD added to
 it; the rectangle contain'd under EA, and
 AD, with the square of CD, or CB, is equal
 to the square of CA; and the angle ABC,
 being a right angle, (as is prov'd in the third
 book) the square of CA is equal to the squares
 of AB and BC; therefore the rectangle un-
 der AE and AD, with the square of BC, is
 equal to the squares of AB and BC. Take
 therefore from them both the square of BC,
 and the rectangle under AE, and AD, will be
 equal to the square of AB. Divide therefore
 the known square of AB, by the hight of the
 mountain AD, and the Quotient will be the
 line AE; from which subtracting the hight



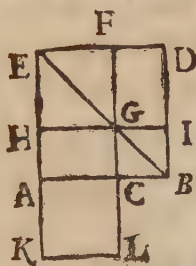
of the mountain, the remainder will be the diameter of the earth DE.

We have made use likewise of the same proposition in our *Algebra*, to demonstrate the thirteenth proposition of the third Book, to find the root of a square equal to a number, more or less than a certain number of roots. The two that follow do also serve for the proof of the like operations.

PROPOSITION VII.

A THEOREM.

If a line be divided, the square of the whole line, with that of one of its parts, is equal to two rectangles contain'd under the whole line, and that first part, together with the square of the other part.



LET the line AB be divided in any where in C; the square AD of the line AB, with the square AL, will be equal to two right angles contain'd under AB and AC, with the square of CB. Make the square of AB, and having drawn the diagonal EB, and the lines CH and HGI; prolong EA so far, as that AK may be equal to AC: so AL will be the square of AC, and HK will be equal to AB; For HA is equal

The Second Book.

III

GC, and GC is equal to CB, because CI is the square of CB, (*by the Corol. of the 4.*)

Demonstration.

'Tis evident, that the squares of AD and AL are equal to the rectangles HL and HD, and the square CI. Now the rectangle HL is contain'd under HK equal to AB, and KL equal to AC. In like manner the rectangle HD is contain'd under HI equal to AB, and HE equal to AC. Therefore the squares of AB and AC are equal to two rectangles contain'd under AB and AC, and the square of CB.

In Numbers.

Suppose the line AB to consist of 9 parts, AC of 4, and CB of 5. The square of AB 9 is 81, and that of AC 4 is 16; which 81 and 16 added together make 97. Now one rectangle under AB and AC, or 4 times 9, make 36, which taken twice is 72: and the square of CB 5 is 25; which 72 and 25 added together make also 97.

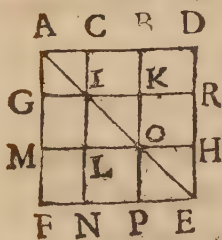
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PROP.

PROPOSITION VIII.

A THEOREM.

If you divide a line, and add another to it equal to one of its parts, the square of the whole compounded line will be equal to four rectangles contain'd under the first line, and that part that is added, together with the square of the other part.



LET the line AB be divided any where in the point C, and BD equal to CB added to it: the square of AD will be equal to four rectangles contain'd under AB, and BC or BD, and the square of AC. Make the square of AD, and having drawn the diagonal AE, draw likewise the perpendiculars BP, and CN, cutting the diagonal in I, and O: and also the lines MOH, and GIR, parallel to AB. The rectangles GC, LK, PH, MB, and NR, will be squares, (by the Coroll. of the 4.)

Demonstration.

The square ADEF is equal to all its parts, and the rectangles LB, OD, PM, are contain'd under lines equal to AB, and BC, and if you add the rectangle MI to the rectangle PH, they together will give you another rectangle contain'd under

under a line equal to AB, and another equal to CB or BD. Besides which there remains nothing but the square GC, which is the square of AC. Therefore the square AD is equal to four rectangles contain'd under AB and BD, and the square of AC.

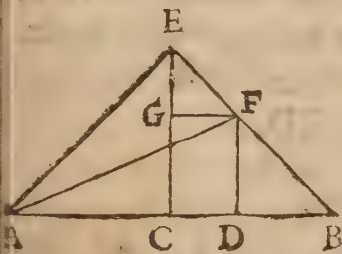
In Numbers.

Let the line AB consist of 7 parts, AC of 3, and CB of 4; as also BD: the square of AD 11 will be 121. And one rectangle under AB 7, and BD 4, makes 28; which taken four times is 112; and those together with the square of 3, which is 9, make also 121.

PROPOSITION IX.

A THEOREM.

If a line be divided into two equal parts, and two unequal, the squares of the unequal parts will be double the square of half the line, and the square of the intermediate part.



LET the line be divided into two equal parts at the point C, and two unequal at the point D: the squares of the unequal parts AD, and DB will,

will be double the squares of AC, which is half AB, and CD the intermediate part. Draw CE perpendicular to AB, and equal to AC; draw also the lines AE, and BE, and the perpendicular DF, as likewise FG parallel to CD. Then joyn the line AF.

Demonstration.

The lines AC and CE are equal, and the angle C is a right angle: therefore (*by the 5.1.*) the angles CAE, and CEA, are equal; and consequently half right angles. In like manner, the angles CEB, CBF, GFE, and DFB, are half right angles; and the lines GF and GE, DF and DB, equal, (*by the 6.1.*) and the whole angle AEF is a right angle. Now the square of AE (*by the 47.1.*) is equal to the squares of AC and CE, which are equal: therefore it is double the square of AC. For the same reason, the square of EF is double the square of GF, or CD. Now the square of AF is equal to the squares of AE, and EF, because the angle AEF is a right angle: therefore the square of AF is double the squares of AC, and CD. The same square of AF is likewise equal to the squares of AD, and DF or DB, the angle D being a right angle. Therefore the squares of AD, and DB, are double the squares of AC, and CD.

In Numbers.

Let AB be 10, AC 5, CD 3, and DB

2 : the squares of AD 8, and DB 2, that is to say, 64 and 4, which make 68, are double the squares of AC 5, that is, 25, and of CD 3, which is 9 : for 25 and 9 make 34, which is half of 68.

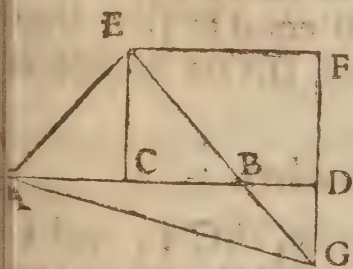
The USE.

"I have not met with this Proposition, except in *Algebra*; no more than that which follows.

PROPOSITION X.

A THEOREM.

If a line be added to another that is divided into two equal parts; the square of the line compounded of those two, with the square of that which is added, makes double the square of half the line, and the square of that which is compounded of half, and the line that is added.



LET the line AB be divided in the middle at the point C, and the line BD added to it: the squares of AD, and BD, will be double the squares of the perpendiculars CE and DF equal to AC: and then draw the lines AE,

AC, and CD. Draw
and DF equal to AC:

AE, EF; and producing FD to G, so that DG may be equal to BD, joyn the lines AG, and EBG.

Demonstration.

The lines AC, CB, and CE being equal, and the angles at the point C being right angles: the angles CAE, AEC, CEB, and CBE, will be half right angles. In like manner the angle DD being a right angle, and the lines BD and DG equal, the angles DBG, and DGB, will be half right angles; and so will likewise GEF, the angle F being a right angle; therefore the lines FG and FE are equal, (*by the 6. 1.*) and EF is equal to CD, (*by the 33. 1.*) Now the square of AE is double the square of AC, and the square of EG also double the square of EF, and CD, (*by the 47. 1.*) But the square of AG is equal to the squares of AE and EG, (*by the same*;) therefore the square of AG is double the squares of AC and CD. The same square of AG is likewise (*by the same*) equal to the squares of AD, and DG equal to DB: therefore the squares of AD and DB are double the squares of AC and CD.

By Numbers.

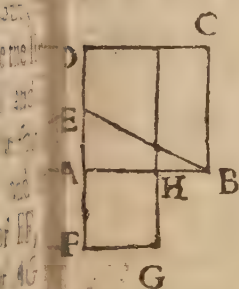
Let AB contain 6 parts, AC 3, and CB 3, BD 4; the square of AD 10 is 100: the square of BD 4 is 16, which make together 116. The square also of AC 3 is 9: the square

square of CD 7 is 49. Now 49 and 9 make 58, the half of 116.

PROPOSITION XI.

A PROBLEM.

To divide a line in such a manner, that the rectangle under the whole line, and one of its parts, shall be equal to the square of the other part.



Suppose the line AB to be divided in such a manner, that the rectangle under the whole line AB, and BH, may be equal to the square of AH. Make the square of AB, (by the 46. 1.) and dividing AD in the middle in E, draw EB, and take EF equal to EB. Then make the square of AF, that is to say, let AF and AH be equal. I say, the square of AH will be equal to the rectangle HC, contain'd under HB, and AC equal to AB.

Demonstration.

The line AD is divided equally in the point E, and the line FA added to it, therefore (by Prop. 6.) the rectangle DG contain'd under DF, and FG equal to AF, with the square of AE, is equal to the square of EF, equal to EB: now the square of EB is equal to the squares of AE and AB

Q

118 *The Elements of Euclid.*

AB, (by the 47.1.) therefore the squares of AB and AE are equal to the rectangle DG, and the square of AE: and subtracting from both the square of AE; the square of AB, that is, AC, will be equal to the rectangle DG: taking away therefore the rectangle DH, which is common to both, the rectangle HC will be equal to the square of AH, that is, AG.

The use.

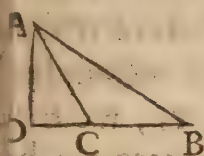
This proposition teaches how to cut a line according to the extreme and middle proportion, as will be shown in the 6th. Book, This is also frequently made use of in the 14th. book of *Euclid's Elements*, to find the sides of regular Solids. It is useful also in the 11. of the 4. to inscribe a Pentagone in a circle, as also a Pentedecagone (or a figure with 15. angles.) You will see also other uses thereof in dividing lines on this manner, in the 30th. proposition of the 6.

PROP.

PROPOSITION XII.

A THEOREM.

In an Obtuse angle triangle, the square of the side oppos'd to the obtuse angle, is equal to the squares, of both the other sides, and two rectangles contain'd under the line upon which a perpendicular will fall, and the line which lies betwixt the triangle and the perpendicular.



LET the angle ACB, of the triangle ABC, be an obtuse, and let AD be drawn perpendicular to BC; the square of the side AB is equal to the squares of the sides AC and CB, and two rectangles contain'd under the side BC, and DC.

Demonstration.

The square of AB is equal to the squares of AD, and DB, (*by the 47.1.*) But the square of DB is equal to the squares DC, and CB, and two rectangles contain'd under DC and CB, (*by the 4.*) Therefore the square of AB is equal to the squares AD, DC, and CB, and two rectangles contain'd under DC and CB. In stead of the two first squares AD, and DC, put the square of AC, which is equal to them, (*by the 47.*) The square AB will be equal to the squares

Q 2

AC

AC and CB, and two rectangles contain'd under DC and CB.

The use.

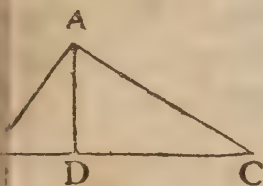
' This proposition is useful in *Trigonometry*
' to measure the *area* of a triangle, whose sides
' are known. For example, suppose the side
' AB to consist of twenty foot, AC of 13, BC
' of 11: the square of AB will be 400, that of
' AC 169, and that of BC 121. The summe
' of the two last is 290, which subtracted from
' 400, there will remain 110 for the two rect-
' angles under BC and CD. The half of which,
' 55, will make one half of those rectangles;
' dividing which number by BC, 11, we shall
' have 5 for the line CD; whose square 25 be-
' ing subtracted from the square of AC, 169,
' leaves the square of AD, 144, whose root 12
' will be the side AD; which being multiplied
' by $5\frac{1}{2}$, the half of BC, will give the *area* of
' the triangle ABC, that is, 66 foot square.

PROP.

PROPOSITION XIII.

A THEOREM.

In any triangle whatsoever, the square of the side oppos'd to the acute angle, with two rectangles contain'd under the side upon which the perpendicular falls, and the line which is betwixt the perpendicular and that angle; is equal to the squares of both the other sides.



Suppose the triangle to be ABC , and the acute angle C , and AD the perpendicular falling upon BC : the square of the side AB , oppos'd to the acute angle C , with two rectangles contain'd under BC and CD , will be equal to the squares of AC , and BC .

Demonstration.

The line BC being divided in D , (*by the 7.*) the squares of BC and DC are equal to two rectangles under BC and CD , and the square of BD . Add to both the square of AD : the squares of BC, DC , and AD , will be equal to two rectangles under BC , and CD , and the squares of BD , and AD . In stead of the squares of CD , and AD , put the square of AC , which is equal to them, (*by the 47. 1.*) and instead of the

the squares of BD, and AD, substitute the square of AB, which is equal to them, (*by the same:*) the squares of BC, and AC, will be equal to the square of AB, and two rectangles contain'd under BC, and CD.

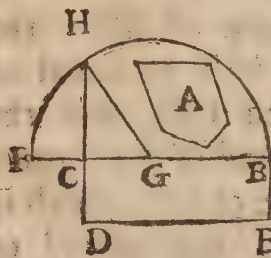
The U S E.

‘ These propositions are very useful in *Trigo-*
‘ *nometry*: I have made use of them in the
‘ eighth proposition of my third book, to prove,
‘ that in a triangle the Sine total has the same
‘ proportion to the sine of an angle, as the rect-
‘ angle contain’d under the sides, which form
‘ that angle, to double the triangle. I have
‘ us’d them likewise in the seventh, and divers
‘ other propositions.

PROPOSITION XIV.

A. PROBLEMS.

To describe a square equal to a rectilineal figure
given,



TO describe a square equal to the rectilineal A, make (by the 45.1.) a rect. angle BCDE equal to the rectilineal A. If the sides CD, and CB were equal, we should

should have what we desir'd : but being unequal, continue the line BC, so that CF may be equal to CD ; and dividing the line BF, in the middle at the point G, describe the semicircle FHB ; this done, prolong DC to H. The square of CH is equal to the rectilineal A. Draw the line GH.

Demonstration.

The line BF is divided into two equal parts in G, and two unequal in C : therefore (*by the 4. 1.*) the rectangle contain'd under BC, CF, or CD, that is to say, the rectangle BD, with the square of CG, is equal to the square of GB, or GH, which is equal to it. Now (*by the 47. 1.*) the square of GH is equal to the squares of CG, and CH : therefore the rectangle BD, and the square of CG, are equal to the squares of CG, and CH ; and therefore taking away the square of CG, which is common to both, there will remain the square of CH equal to the rectangle BD, or, which is the same, the rectilineal A.

The Use.

' This proposition teaches us in the first place to reduce any rectilineal figures to squares ; which being the chief measure of all superficies's, because its dimensions are both equal, we can by this means take the magnitude of all sorts of rectilineal figures. Again it helps us to find a middle proportional betwixt two lines

‘ lines given, as we shall see in the thirteenth
 ‘ proposition of the sixth book.

‘ *Aristotle* brings this proposition as an in-
 ‘ stance of a Formal Definition: for, in his second
 ‘ book, *de Anima*, sect. 12. distinguishing be-
 ‘ twixt a Formal and a Causal Definition, he ex-
 ‘ plains them thus. If, when tis demanded
 ‘ What it is to square a Rectangle? answer be:
 ‘ return’d, that it is to describe a square equal
 ‘ to a rectilineal; this answer contains the for-
 ‘ mal definition. But if it be said, that it is to
 ‘ find a middle proportional betwixt two lines;
 ‘ this gives the Causal definition. For to find
 ‘ a middle proportional is the cause of making
 ‘ a square equal to the rectilineal propos’d.

‘ This proposition may also be farther useful
 ‘ for the squaring of crooked figures; and also,
 ‘ as far as is possible, even the Circle it self; for
 ‘ all sorts of crooked figures may, at least as far
 ‘ as is discernible by sense, be reduc’d to recti-
 ‘ lineals. As for example, if we inscribe in a
 ‘ circle a Polygone consisting of a thousand
 ‘ sides, there will be no sensible difference be-
 ‘ twixt it and the circle: therefore reducing
 ‘ this Polygone to a square, we do, as far as our
 ‘ senses are capable of judging, square the circle

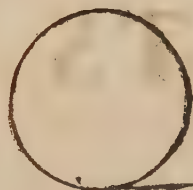
THE THIRD BOOK
OF THE
ELEMENTS
OF
EUCLID.

THis third Book explains the properties of a Circle, and compares divers lines which may be drawn within, or without its circumference. It considers likewise the circumstances of circles, that cut each other. touch a right line; and the differences of angles that are made either at the centers or circumferences. In fine it lays down the first principles for the establishing the practical part of *Geometry*; for which the circle is most commodiously made use of in almost all treatises of the *Mathematicks*,

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DEFINITIONS.

1. Those Circles are equal whose diameters or semidiameters are equal.



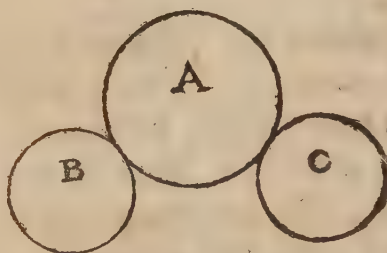
A

B

2. A line is said to touch a circle, when, meeting with its circumference, it does not cut it.

the line AB.

3. Circles touch when meeting, they do not cut each other. the circles A, B, and C.



A

C

F

E

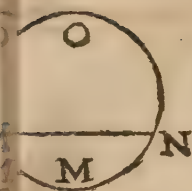
G

B

D

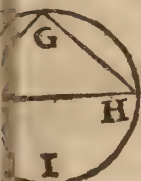
4. Those lines are equally remote from the center; when the perpendiculars, drawn from the center to the lines, are equal. "As for example, if EF, and EG, perpendiculars to the lines AB, and CD

be equal, AB and CD will be equally remote from the center; because the distance of a line from the center is always to be measur'd by perpendicular line.



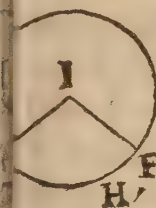
5. A segment of a circle is a figure terminated on one side by a right line, and on the other by the circumference of a circle. As LON, LMN.

6. The angle of the segment is the angle which the circumference makes with the right line. "As the angles LNO, NLM.



7. An angle is in that segment in which are the lines that form it. "As the angle FGH, is in the segment FGH.

8. An angle is upon that arch, to which it is oppos'd, or which is as its base. "As the angle FGH, is upon the arch FIH.

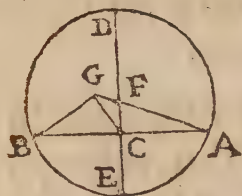


9. The Sector is a figure contain'd under two semidiameters, and the arch which serves them for a base. "As the figure FGH.

PROPOSITION I.

A PROBLEM.

To find the center of a Circle.



TO find the center of the circle AEBD, draw the line AB, and divide it in the middle at the point C; through which draw the perpendicular ED, which also divide into two equal parts at the point F, and that point F will be the center of the circle. If it be not, suppose the point G be the center; and draw the lines GA, GB, and GC.

Demonstration.

If the point G be the center, the triangles GAC, and GBC, will have the sides GA, and GB equal, (*by the definition of a circle*;) and AC and CB will be equal, the line AB being divided in the middle at the point C, and CG being common, the angles GCB, and GCA will be equal, (*by the 8. 1.*) and CG a perpendicular, to CD, which is contrary to the supposition. Therefore the center must of necessity be in the line CD. I add, that it must be at the point F, where it is divided into two equal parts: otherwise

wise the lines drawn from the center to the circumference would not be equal.

Coroll. The center of a circle is in that line, which falling perpendicularly upon another, divides it into two equal parts.

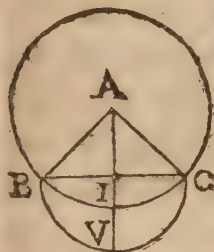
The USE.

“This proposition is necessary to demonstrate those that follow.

PROPOSITION II.

A THEOREM.

A right line drawn from one point of the circumference to another, falls wholly within the circle.



LET a line be drawn from the point B to the point C. I say, it will be wholly contain'd within the circle. To prove that it cannot fall without the circle, as BVC; having found the center of the circle A, draw the lines AB, AC, and AV.

Demonstration.

The sides AB, and AC, of the triangles ABC, are equal: therefore (by the 5. 1.) the angles ABC and ACB are equal, And since the angle AVC is an external angle in respect of the triangle

angle AVB, it is greater than the angle ABC, (*by the 16. 1.*) and then also it will be greater than the angle ACB. Therefore (*by the 19. 1.*) in the triangle ACV, the side AC, oppos'd to the greater angle AVC, will be greater than AV: and by consequence AV ought not to reach to the circumference of the circle, if the line BVC was a right line.

The U S E.

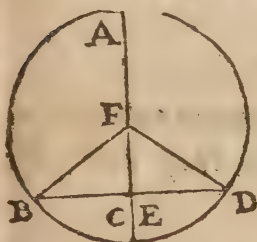
'Tis by this proposition that they demonstrate, that a circle can touch a right line but in one place. For if the line touch'd two points of the circumference, it would be drawn from one of its points to another: and by consequence, according to this proposition, would enter the circle; though by its definition, the line that touches ought not to cut the circumference. *Theodosius* makes use of the same Demonstration to prove, that a Globe can touch a plane only in one point; for otherwise the plane would enter within the Globe.

PROP.]

PROPOSITION III.

A THEOREM

If the Diameter divide a line, which does not pass through the center, into two equal parts, it will cut it at right angles; and if it cut it at right angles, it will divide it into two equal parts.



IF the diameter AC, cut the line BD, which does not pass through the center F, into two equal parts at the point E, it will cut it at right angles. Draw the lines FB, and

FD.

Demonstration.

In the triangles FEB, and FED, the side EF is common; the sides BE and ED are equal, because the line BD is equally divided in E, and their bases FB and FD are equal: therefore (by the 8.1.) the angles BEF and DEF are equal, and by consequence right angles. I add, that if the angles BEF and DEF be right angles, the line BD will be divided into two equal parts at E, that is to say, the lines BE and ED will be equal.

Demonstration.

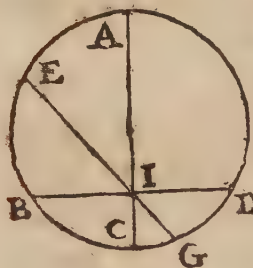
The triangles BEF and DEF are rectangular: therefore (by the 47. 1.) the square of the side

side DF will be equal to the squares of the sides ED, and EF. Now the squares of BF and FD are equal, because the lines are equal, therefore the squares of BE and EF are equal to the squares of DE and EF; and taking away the square of EF, the squares of BE, and ED will be equal, and by consequence the lines.

PROPOSITION IV.

A THEOREM.

Two lines drawn within a circle cannot cut each other into two equal parts, unless they both pass through the center.



IF the lines AC and BD cut each other at the point I, which is not the center of the circle, they will not equally divide each other. First, if one of those lines, as AC, pass through the center, 'tis evident it cannot equally be divided but at the center. But if neither pass through the center, as BD and EG, draw the line AIC through the center.

Demonstration.

If the line AC divide the line BD into two equal parts in I, the angles AID and AIB will be

right angles, (by the 3.) In like manner if the line EG was equally divided in I, the angle IE would be a right angle; and consequently the angle AIB and AIE would be equal, which is impossible, one being part of the other. In a word the line AIC, which passes through the center, would be perpendicular to the lines AD and EG, if they were both equally divided at the point I.

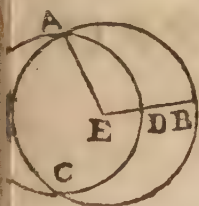
The use.

These two Propositions are us'd in *Trigonometry*, to demonstrate, that the half of a chord of an arch is perpendicular to the semidiameter; and consequently, that it is the sine of half the arch. By these also they demonstrate, that the sides of a triangle have the same proportion, as the sines of the opposite angles. We also make use of it to find the Eccentricity of the Circle, which the Sun describes in his annual motion,

PROPOSITION V.

A THEOREM.

Circles that cut each other, have not the same center.



THE circles ABC, and ADC, which cut each other in A and C, have not the same center. If they had the same center,

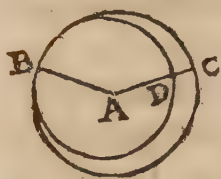
S. Sup

suppose E, the lines EA and ED would be equal, (*by the definition of a circle*;) as also the lines EA, and EB: therefore the lines ED and EB would be equal, which is impossible, one being part of the other.

PROPOSITION VI.

A THEOREM.

Two circles that touch each other on the inner sides have not the same center.



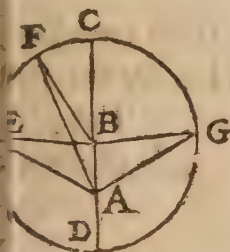
THE circles BD and BC,, which touch each other on the inner side at the point B,, have not the same center. For should the point A be suppos'd to be the center of both the circles; the lines AB and AC, AB and AD, would be equal, (*by the definition of a circle,*) and consequently the lines AD and AC would be equal, which is impossible, one being part of the other.

PROP.

PROPOSITION VII.

A THEOREM.

many lines be drawn from any one point within the circle, which is not its center, to the circumference: 1. that which passes through the center is the greatest: 2. the remainder of it, continu'd to the opposite part of the circumference, is the least: 3. that which is nearest to the greatest, exceeds those that are more remote: 4. There can be no more than two of them equal to each other.



Suppose many lines to be drawn from the point A, being not the center of the circle, to the circumference; and the line AC to pass through the center B: I will demon-

strate, that it is greater than any of the other; for example, that it is greater than AF. Draw FB.

Demonstration.

The sides AB and BF of the triangle ABF, are greater than AF alone, (by the 20.1.) But BF and BC are equal, (by the definition of a circle:) therefore AB and BC, that is to say, AC, is greater than AF.

I add in the second place, that AD is the least;

S 2

least; for example, that it is less than AEB.
Draw BE.

Demonstration.

The sides EA and AB are greater than BB alone, but BE is equal to BD, therefore EA and AB are greater than BD: taking therefore from both that which is common AB, AE will remain greater than AD.

Further, AF, which is nearer AC than AEB is also greater than it.

Demonstration.

The triangles FBA, and EBA, have the sides BF and BE equal, and BA is common to both but the angle ABF is greater than the angle ABE: therefore (by the 24. 1.) AF is greater than AE.

Lastly, I say, that no more than two lines that are equal to each other, can be drawn from the point A to the circumference. Take the angles ABE and ABG equal; and draw the lines AE and AG.

Demonstration,

The triangles ABG, and ABE, having the sides BE and BG equal; the side AB common to both, and the angles ABE and ABG equal; therefore their bases AE, and AG will be equal (by the 4. 1.) But all the lines that can be drawn either on one side or the other, will be either nearer AC, than AE, and AG; or more remote from it; and accordingly will be either greater or less than AG. Therefore there can no

more

more than two lines equal betwixt themselves
be drawn from the point A to the circumfe-
rence.

The N S E.

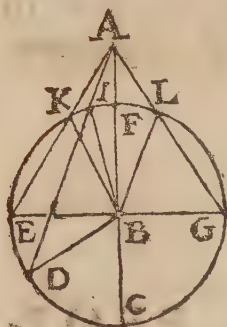
Theodosius advantageously uses this propo-
sition to prove, that if from any point of the su-
pericies of a sphere, which is not the pole of
any certain circle, divers arches of greater
circles be drawn to the circumference of that
circle, that which passes through its pole will
be the greatest. For example: If from the pole
of the world, which is distinct from the pole of
the Horizon, (for the Zenith is its pole,) divers
arches of greater circles be drawn to the cir-
cumference; the arch of the Meridian, which
passes through the Zenith, will be the greatest
arch. This proposition is also brought to
prove, that the Sun, when in his *Apogæum*,
is most remote from the Earth:

PROP-

PROPOSITION VIII.

A T H E O R E M.

If from a point taken without the circle, many lines be drawn to its circumference, 1. of all those that extend to the concave circumference, that which passes through the center is the greatest: 2. those that lye nearest to it, are greater than those that are more remote: 3. among those that fall upon the convex circumference, that which being continu'd passes through the center, is the least: 4. the nearer to that are less than those farther off: 5. there can be but two equal lines drawn from the same point either to the concave or convex circumference.



SUPPOSE many lines were drawn from the point A to the circumference of the circle GCDE.

First, the line AC, which passes through the center B, is the greatest of all those that reach to the concave circumference; for example, it is greater than AD. Draw the line BD.

Demonstration.

In the triangle ABD, the sides AB and BD are greater than AD alone; but the sides AE

and

and BC are equal to AB and BD : therefore AB and BC, or AC, is greater than AD.

2. AD is greater than AE.

Demonstration.

The triangle ABD and ABE, have the side AB common to both, and the sides BD and BE equal, and the angle ABD is greater than the angle ABE: therefore (*by the 24. 1.*) the base AD is greater than the base AE.

3. AF, which being continu'd passes through the center, is the least of all those that are drawn to the convex circumference LFIK; for example, it is less than AI. Draw IB.

Demonstration.

In the triangle AIB the sides AI and IB are greater than AB alone, (*by the 20. 1.*) therefore taking from both the equal lines BI and BF, AF will remain less than AI.

4. AI is less than AK. Draw the line BK.

Demonstration.

In the triangles AIB and AKB, the sides AK and KB are greater than the sides AI, and IB, (*by the 21. 1.*) therefore taking from both the equal sides BK, and BI, AI will remain less than AK.

5. There can be but two lines equal betwixt themselves drawn. Take the angles ABL, and ABK; as also ABE, and ABG, equal.

Demonstration.

The triangles ABL, and ABK, will have their bases

bases AL and AK equal, (*by the 4. 1.*) and by the same also AE and AG will be equal ; but no other line can be drawn, that will not be either nearer to, or more remote from AF, or AG ; and consequently, that will not be either greater or less than AK and AL, AE and AG.

PROPOSITION IX.

A THEOREM.

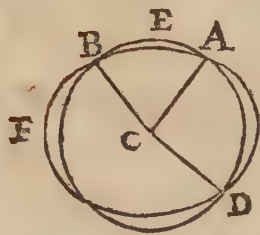
That point from whence three equal lines can be drawn to the circumference of a circle, is its center.

IF the point were not the center of a circle there could be but two equal lines drawn from it to the circumference, (*by the 7, and 8.*)

PROPOSITION X.

A THEOREM.

Two circles cut each other only in two points.



IF two circles AEBD, and ABFD, should cut each other in three points A, B, and D ; find (*by the 1.*) the center C of the circle AEBD ; and draw the lines CA, CB, and CD.

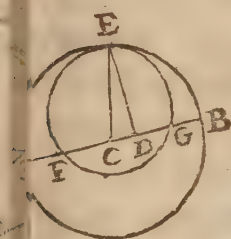
Demonstration.

The lines CA, CB, and CD, drawn from the center C to the circumference of the circle EBD, are equal : but the same lines are also drawn to the circumference of the circle ABFD: therefore (by the 9.) the point C will be the center of the circle ABFD. So that two circles, which cut each other, will have the same center; which is contrary to the fifth proposition.

PROPOSITION XI.

A THEOREM.

Two circles touch each other on the inside, a line drawn through both the centers, will also pass through the point where they touch.



IF the two circles EAB and EFG touch each other on the inside, at the point E; a line drawn through both their centers will pass through the point E. For if the point D is the center of the lesser circle, and C that of the greater, so that the line CD passing through both should not pass through the point E, draw the lines CE and DE.

Demonstration.

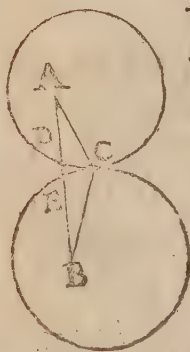
The lines DE, and DG, drawn from the
T center

center of the lesser circle D to its circumference, would be equal: and adding the line CD, the lines ED, and DC, would be equal to CG. Now ED and DC are greater than EC alone, (*by the 20. 1.*) and so CG will be greater than CE: yet C being the center of the greater circle, CE and CB are equal: therefore CG will be greater than CB, which is impossible.

PROPOSITION XII.

A THEOREM.

If two circles touch each other on the outside, a line drawn through both their centers, will pass through the point where they touch.



IF the line AB, which does not pass through the point C where the circles touch, be said to be drawn from the center A to the center B, draw the lines AC and BC.

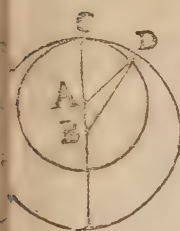
Demonstration.

In the triangle ACB, the side AC and BC would not be greater than the side AB alone, (which is contrary to the 20. 1.) because AD and AE, as also BE and BC, are equal.

PROPOSITION XIII.

A THEOREM.

Two circles can touch each other only in one point.



First, if two circles touch each other on the inside, they will touch but in one point only, the point C; which is marked out by the line BAC passing through both their centers, A, and B. For if

they should touch likewise in the point D, draw the lines AD, BD.

Demonstration.

The lines AC and AD, drawn from the center of the lesser circle to its circumference, are equal: and adding AB, the lines BA, AC, and A and AD, would be equal. Now BC and BD, drawn from the center of the greater circle to its circumference, will be equal: therefore the sides BA and AD will be equal to the side BD alone, which is contrary to the 20. 1.



Secondly if two circles touch each other on the outside; drawing the line AB from one center to the other,

it will pass through the point C, where the circles touch, (*by the 12.*) But if you say that they touch also at the point D: having drawn the lines AD and BD; the lines BC and BD, AC and AD, being equal, the two sides of a triangle taken together, would be equal to the third, which is contrary to the 20. 1.

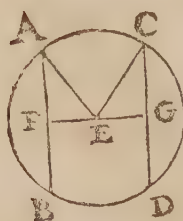
The U S E.

‘ These four propositions are very clear, and evident; and also necessary in *Astronomy*, when we make use of *Epicyles*, to explain the motions of the *Planets*.

PROPOSITION XIV.

A THEOREM.

Equal lines drawn within a circle, are equally remote from the center; and those that are equally remote from the center, are equal.



Supposing the lines AB and CD to be equal: I prove, that the perpendiculars EF and EG, drawn from the center, are also equal. Draw the lines EA and EC.

Demonstration.

The perpendiculars EF and EG divide the lines AB and CD in the middles at the points

F and G, [by the 3.] therefore AF and CG are equal. The angles F and G are right angles: therefore [by the 47. 1.] the square of EA is equal to the squares of EF and FA; as also the square of EC is equal to the squares of EG and GC: but the squares of EA and EC are equal, because the lines EA and EC are equal: therefore the squares of EF and FA are equal to the squares of EG and GC: and taking away the equal squares AF and CG, there will remain the squares of EF and EG equal; and consequently the lines EF and EG, which are the distances of the lines AB and CD from the center, are equal.

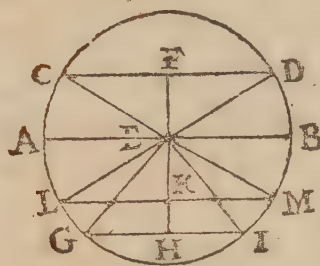
But supposing the distances or perpendiculars EF and EG to be equal; I will prove after the same manner that the squares of EF and FA are equal to the squares of EG and GC; and taking away the equal squares of EF and EG, there will remain the squares of AF and CG equal. And therefore the lines AF and CG, and their double AB and CD, are equal.

PROP.

PROPOSITION XV.

A THEOREM.

The Diameter is the greatest of all lines inscribed in a Circle; and of the rest that is the greatest which is nearest the center.



THE diameter AB is the greatest of all lines that can be drawn in the circle GIDC. As for example, it is greater than CD; for draw the lines EC and ED

Demonstration.

In the triangle CED, the sides EC and ED are greater than CD alone, [by the 20.1.] but AE and EB, or AB, is equal to EC and ED; therefore the diameter AB is greater than CD.

Secondly, let the line GI be more remote from the center than the line CD; that is to say, let the perpendicular EH be greater than the perpendicular EF. I say that CD is greater than GI. Draw the lines EC, and EG.

Demonstration.

The squares of CF and FE [by the 47.1.] are equal to the square of EC: but the square of EC is equal to the square of EG, and the square of EC

EG equal to the squares of GH and HE : therefore the squares of CF and FE are equal to squares of GH and HE ; and taking from one side the square of HE, and from the other the square of EF, which is less than the square of HE, the square of CF will remain greater than the square of GH. Therefore the line CF will be greater than the line GH ; and the whole line CD, the double of CF, will be greater than GI, the double of GH.

The USE.

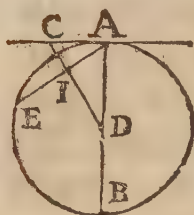
'*Theodosius* makes use of these two Propositions to demonstrate, that in a sphere the lesser circles are more remote from the center. I have also made use of them in *Astrolabes*. To these Propositions may likewise be referr'd that Mechanical proposition of *Aristotle* , by which he shows, that the Rowers at the middle of a Gally have greater force, than those that are at, either the fore, or hinder part thereof ; because the sides of the Gally being crooked, the Oars of the middle part are longer, *i. e.* reach farther, than the rest. The Demonstrations relating to the *Iris*, or Rainbow, do also suppose the truth of these propositions.

THE

PROPOSITION XVI.

A THEOREM.

A line drawn perpendicularly upon the extremity of the diameter falls wholly on the outside of the circle, and touches it. But any other line drawn betwixt that and the circumference of the circle, enters within the circle, and cuts it.



LET the perpendicular AC be drawn upon the point A, which is the extremity of the diameter AB: I say first, that all the other parts of the same line, for example the point C, fall on the outside of the circle. Draw the line DC.

Demonstration.

Since the angle DAC of the triangle DAC is a right angle, DCA will be an acute: and (by the 19. 1.) the side DC will be greater than the side DA; therefore the line DC reaches beyond the circumference of the circle.

I add, that the line CA touches the circle, because that meeting with it at the point A it does not cut it, but all its points are on the outside of the circle.

I say also that no other line can be drawn from

From the point A below CA, which does not touch the circle. If there could, suppose EA to be such an one; and from the point D draw a perpendicular to it, DI.

Demonstration.

Since the angle DIA is a right angle, and the angle IAD an acute, AD will be greater than DI: therefore the line DI does not reach to the circumference, but the point I is within the circle.

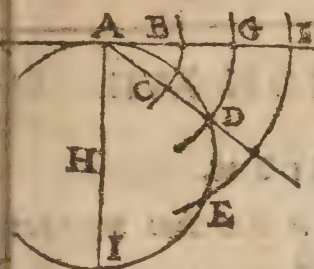
The U S E.

Some Philosophers use this Proposition, but together in vain, to prove, that quantity is not divisible in infinitum, or that there really is in the world such things as Zenonical, i.e. absolutely and in their own nature indivisible points. For the proposition does not, as they should have it, prove, that a circle touches a right line in a Zenonical, but in a Mathematical point, which is nothing else but a quantity consider'd without distinction of parts, that is to say, without conceiving them distinct and separate one from the other, whether in reality it has such parts or not making no matter. We can therefore take any quantity whatsoever for a Mathematical point; which being once establish'd, our Circle will consist of such points, and will be mathematically perfect, provided it touch not a right line, but in a point equal to that quantity which we have taken

ken for a point. But if we afterwards take
 less part for our *Mathematical point*, the circle
 which was exactly perfect according to the
 first supposition, will be imperfect in the second,
 and degenerate into a Polygone. I believe, tis as
 impossible to describe a circle that according to any
 supposition whatsoever shall be most exactly perfect,
 as it is to conceive the least possible quantity.

Secondly, those consequences, which some men draw from this proposition relating to the angle of contact, which they take to be less than any rectilineal angle, are grounded upon this mistake, that they imagine an angle to be a true quantity; the contrary of which may appear from hence, That the lines, that contain an angle, being produc'd to any longitude, the angle becomes not at all the greater. Further, it ought to be duly consider'd, when we mean, when we say, that one angle is greater than another; for this is all we understand, that a circle being described from the point of concurrence at any distance whatsoever the lines of that we call the greater angle will contain betwixt them a greater arch of the circle, than those of that which we call the less; which is the sole meaning of the Excess of one angle above another. From whence we infer, that the angle of contact can no more be compar'd with a rectilineal angle, than a superficies

perfacies with a line, being at the same time both equal, and greater and less than a rectilinear angle. As for ex-



ample; from the point A draw the line AD, making with AE a rectilinear angle; I say it is both greater and less than, and equal to, the angle of contact.

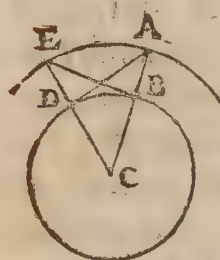
For if we suppose divers circles describ'd from the point A, as the center, whereby to measure those angles: it is evident that, according to the arch drawn beyond the point D, that is the arch EF, the angle of contact is greater than the rectilinear angle. But on the contrary, according to the arch CB, the rectilinear angle is the greater of the two. And lastly, according to the arch DG, passing through the point in which AD cuts the circumference, they are both equal. From whence it follows, that the angle of contact is at the same time both less and greater than, and equal to, the rectilinear angle: and consequently, they ought not at all to be compar'd together. In a word, Angles are no quantities; nor are they call'd less or greater one than another, but with respect to the arches which they contain: so that all the disputes about the angle of contact, and all the Para-

doxes, conclude nothing either for or against the divisibility of quantity; an Angle being no species, but only a property thereof.

PROPOSITION XVII.

A PROBLEM.

From a point given to draw a line that may touch a Circle.



TO draw a line from the point A touching the circle BD, draw the line AC to the center; and at the point B draw a perpendicular BE, which may cut an arch of a circle, described from the center C through the point A, at the point E. Draw also the line EC, and AD. I say the line AD touches the circle in D.

Demonstration.

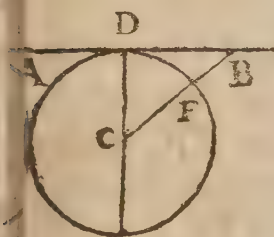
The triangles EBC and ADC have the same angle C; and the sides CD and CB, CE and CA, equal, (by the definit. of a Circle :) and therefore they are equal in all respects, (by the 4. 1.) and the angles CBE and CDA are equal. But the angle CBE is a right angle, therefore the angle CDA will be so too, and (by the 16. 1.) the line AD will touch the circle.

PRO

PROPOSITION XVIII.

A THEOREM.

A line drawn from the center of a circle to the point where a right line touches it, is perpendicular to that line.



IF the line CD be drawn from the center C to the point of contact D, CD will be perpendicular to AB. For if it be not, draw the line CB perpendicular to AB.

Demonstration.

Since the line CB is suppos'd to be perpendicular, the angle B will be a right angle, and consequently CDB an acute, (*by the 32. 1.*) Therefore the line CB, oppos'd to the lesser angle, will be less than CD, which is impossible; because CF, which is but part of CB, is equal to CD.

PROP.

PROPOSITION XIX.

A THEOREM

If a line, perpendicular to the tangent, be drawn from the point of contact, it will pass through the center of the circle.

LET the line AB [see Fig. preced.] touch the circle at the point D, and the line DC be perpendicular to AB. I say, that DC passes through the center. For if it did not, drawing a line from the center to the point D, it would be perpendicular to AB, (by the preceding) and so there would be two perpendiculars drawn to the same point D of the same line, which cannot be.

The USE.

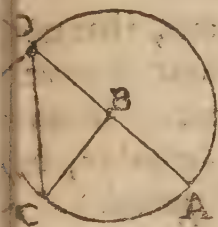
‘The use of lines *Tangents* is very common
 ‘in *Trigonometry*; upon which account it is that
 ‘I have made a table, whereby to measure all sorts
 ‘of triangles, as well spherical as rectilineal. In
 ‘my *Opticks* likewise are divers propositions
 ‘founded upon *Tangents*; as when is determin’d
 ‘what part of a Globe is enlighten’d. The *pha-*
 ‘ses or Apparitions of the Moon are establish’d
 ‘also upon the same doctrine; and that famous
 ‘Problem of *Hipparchus*’s, by which he found
 ‘the distance of the Sun, by the difference of
 the

the true and apparent *Quadratures*. In *Dialling* the *Italian* and *Babylonian* hours are frequently describ'd by lines *Tangents*. Lastly, we take the dimensions of the earth by a line that touches its superficies; and in the art of *Navigation*, take a *Tangent* line for our *Horizon*.

PROPOSITION XX.

A THEOREM.

The angle at the center is double the angle at the circumference, which has the same arch for its base.



IF the angle ABC , which is at the center, and the angle ADC , at the circumference, have the same arch AC for their base, the first will be double the second. This proposition has three

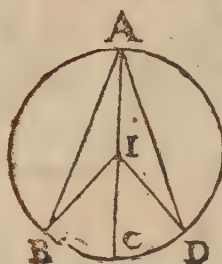
different cases: the first of which is, when the line ABD passes through the center B , the line AB in one triangle concurring with the line BD of the other.

Demonstration.

The angle AEC is the external angle in respect of the triangle BDC : therefore (by the 2. 1.) it is equal to both the angles D and C , which

which being equal, (*by the 5. r.*) because their sides BC and BD are equal, the angle ABC is the double of either.

The second case is, when one angle inclofess the other, but none of the lines that form them concur in one; as you see in the next figure. The

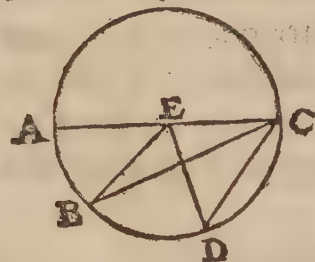


angle BID is at the center, and the angle BAD at the circumference. Draw the line AIC through the center.

Demonstration.

The angle BIC is double the angle BAC, and CID is double the angle CAD, (*by the preceding case* :) therefore the angle BID is double the angle BAD.

The third case is, when it happens, that neither one angle incloses the other, nor does any of the lines that form them, concur in one: *which case is wholly omitted by my Author, but for the Readers satisfaction is here supplied.*



Let the angle at the center be BED, and the angle at the circumference BCD, having the same arch for their base BD. I say, the angle BED is double the angle BCD. Draw the line EC, and continue it to the point A.

Demon-

Demonstration.

The angle AED is double the angle ACD, *the 1. case ;)* and (*by the same*) the angle B is double the angle ACB : therefore the remainder of the one BED is double the remainder of the other BCD.

The USE.

That Problem, which is ordinarily proposed, showing how to describe an *Horizontal* *Sol* by one sole opening of the Compass, is built in part on this Proposition. And again, when we would determine the *Apogæum* of the Moon, or the excentricity of his Circle, by three observations, we suppose the angle at the centre to be double that at the circumference. *Ptolomey* makes frequent use of this proposition to determine both the excentrick circle of the Moon, and the Epicycle of the Moon. The first proposition of the third book of *Trigonometry* is grounded also upon this here.

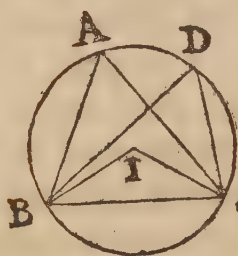
X

PROP.

PROPOSITION XXI.

A THEOREM.

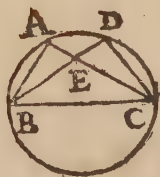
The angles, that are in the same segment of a circle, or that have the same arch for their base, are equal.



IF the angles BAC and BDC are in the same segment of a circle, which is greater than a semicircle, they will be equal. Draw the lines BI and CI.

Demonstration.

The angles A and D are each of them half of the angle BIC, (*by the preceding,*) therefore they are equal. They have likewise the same arch BC for their base.



Secondly, let the angles A and D be in the same segment BAD, which is less than a semicircle; they will nevertheless be equal.

Demonstration.

All the angles of the triangle ABE are equal to all the angles of the triangle DEC, (*by 1. 32. 1.*) but the angles AEB and DEC are equal, (*by the 15. 1.*) Also the angles EAB and EDC

ABE are equal, (*by the preceding case*,) being the same segment ABCD, greater than a semicircle; therefore the angles BAE and BAE are equal; which, the angles at E being equal, and consequently (*by the Coroll. of the 15.*) the lines AE and EC, making but one right angle, as likewise DE and EB another, are the sides A and D, in the same segment ABCD, having the arch BC for their base.

The USE.

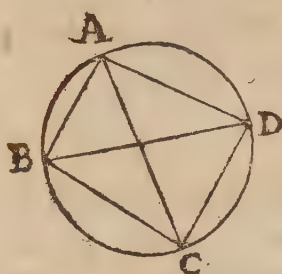
This Proposition is produc'd in *Opticks* to prove, that the line BC will appear of the same greatness, when tis view'd from A, and D, because it is seen in both cases under equal angles.

The same proposition is us'd to describe large circles without having their centers; for example, if we would make large Copper basons of spherical figure, such as we might work upon for polishing Spectacles, and glasses to see at a great distance. For having made in Iron an angle BAC equal to that, which is contain'd in the segment ABC, and at the points B and C strongly fasten'd two small iron pins; if the angle BAC be mov'd so, that the side AB may always touch the pin B, and the side AC the pin C, the point A will describe an arch of the circle ABCD. This manner of describing a circle may also be us'd in making great Globes.

PROPOSITION XXII.

A THEOREM.

Quadrilateral figures, inscrib'd in a circle, have their opposite angles equal to two right angles.



LET a quadrilateral figure, or a figure of four sides, be inscrib'd in a circle, in such sort that all angles may terminate at the circumference of the circle ABCD: I say the opposite angles BAD and BCD are equal to two right angles. Draw the diagonals AC, and BD.

Demonstration,

All the angles of the triangle BAD are equal to two right angles. In stead then of the angle BCD put the angle ACD, which is equal to it, (the 21.) being in the same segment ABCD: instead of the angle ADB, put the angle ACB, which is in the same segment of a circle BADC. Therefore the angles BAD, and the angles ACD and ACB, that is to say, the whole angle BCD, are equal to two right angles.

The USE.

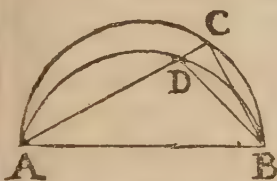
Ptolomey makes use of this Proposition to frame the table of Chords, or lines subtended

ing arches. I have also us'd the same in my third book of *Trigonometry*, to prove, that the sides of an obtusangle triangle would have the same proportion among themselves as the sines of the opposite angles.

PROPOSITION XXIII.

A THEOREM.

Two similar segments of a circle, describ'd upon the same line, are equal.



I call those similar segments of a circle, which contain equal angles; and I say, that if such be describ'd upon the same line AB, they will fall one upon the other, and not exceed each other in any part. For if either did exceed the other, as do the segments ADB, and ACB, they would not be similar: to demonstrate which, draw the lines ADC, BD and BC.

Demonstration.

The angle ADB is an external angle in respect of the triangle DBC: therefore (by the 16. 1.) it is greater than the angle ACB, and by consequence the segments ADB and ACB contain unequal angles, which I say is to be dissimilar.

PROP.

PROPOSITION XXIV.

A THEOREM.

Two similar segments of a circle describ'd upon equal lines, are equal.



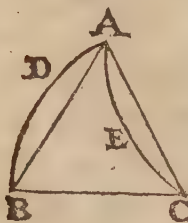
IF the segments of the circles AEB, and CFD, be similar, and the lines AB and CD equal, the segments also will be equal.

Demonstration.

Suppose the line CD to be plac'd upon the line AB, being suppos'd to be equal, they will not exceed each other; and then the segments AEB and CFD will be describ'd upon the same line, and therefore will be equal, (*by the preceding.*)

The USE.

'Crooked figures are frequently reduc'd to
'rectilineals by this proposition. As for ex-

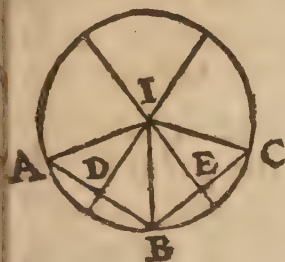


'ample: if two similar segments
'of a circle AEC, and ADB, be
'describ'd upon AB and AC, the
'the equal sides of the triangle
'ABC: tis evident, that, trans-
'posing the segment AEC unto
ADB, the triangle ABC is equal to the figure
ADBCEA. PROP.

PROPOSITION XXV.

A PROBLEM.

To compleat a circle, of which we have but a part.



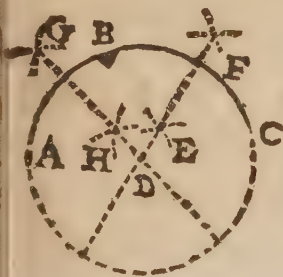
HAVING the arch ABC given, to compleat the circle we must find its center; to which end draw the lines AB and BC, which having divided in the middles at the points E and D, draw their two perpendiculars EI and DI; which will meet at the point I, the center of the circle.

Demonstration.

The center is in the line DI, (*by the coroll. of the 1.*) it is also in EI, (*by the same;*) therefore it must be at the point I.

The USE.

‘ This Proposition occurs very frequently; but sometimes it is express’d in other terms: as to inscribe a triangle in a circle; or to describe a circle through three points given, provided they be not plac’d in a right line. Let the points propos’d be A, B, and C; and placing the foot of the compass at the point C, describe two arches



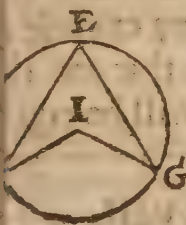
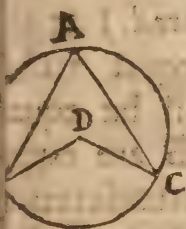
‘ arches F and E, at any distance whatsoever.
‘ Then remove the foot of the compass to the
‘ point B, and at the same distance describe two
‘ other arches cutting the former in E and F ;;
‘ also from the point B, as the center, describe
‘ at any distance the arches G and H, and at
‘ the same distance from the center A two other
‘ arches cutting them in G and H. Which done
‘ draw the lines through F and E, G and H,
‘ which shall cut each other at the point D, the
‘ center of the circle. The Demonstration is
‘ obvious enough; for if you had drawn the
‘ lines AB, and BC, you had, by this opera-
‘ tion, divided them equally and perpendicu-
‘ larly. This proposition is exceeding necessa-
‘ ry to describe Astrolabes, and compleat cir-
‘ cles, of which we have but three points. That
‘ Proposition in *Astronomy*, which teaches how
‘ to find the *Apogæum*, and excentricity of the
‘ circle of the Sun, virtually contains this. And
‘ I also have made frequent use of it in my
‘ Treatise concerning the Cutting of Stones.

PROP.

PROPOSITION XXVI.

A THEOREM.

Equal angles, whether at the centers, or the circumferences of equal circles, have equal arches for their bases.



IF the angles D and I, at the centers of equal circles ABC, and EFG, be equal; the arches BC and FG will be equal. For if the arch BC was greater or less than the arch FG, since the angles are measur'd by arches, the angle D would be greater or less than the angle I.

But if the equal angles be suppos'd to be at the circumferences of equal circles, as A and E;

the angles which they enclose at the centers, as D and I, being their doubles, will be likewise equal, and consequently require equal arches for their bases, BC and FG; which arches are likewise the measures of the angles A and



PROP.

PROPOSITION XXVII.

A THEOREM.

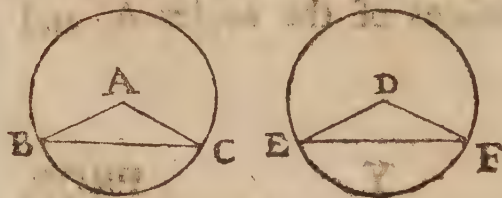
Angles, whether at the centers or circumferences of equal circles, having equal arches for their bases, are also equal.

IF the angles D and I (fig. preced.) at the centers of equal circles have equal arches BC and FG for their bases, they will be equal, because their measures BC and FG are equal. And if the angles A and E, at the circumferences of equal circles have equal arches BC and FG for their bases, since the angles they enclose at the centers will be equal, they also that are the halves of those angles (by the 20.) will be equal.

PROPOSITION XXVIII.

A THEOREM.

Equal lines, within equal circles, answer to equal arches.



IF the equal lines BC and EF be applied to equal circles, ABC and DEF, and

and DEF, they will be the chords of equal arches, BC, and EF. Draw the lines AB, AC, ED, EF.

Demonstration.

In the triangles ABC and DEF, the sides AB and AC, DE and DF are equal, being the semi-diameters of equal circles: and their base's BC and EF are suppos'd equal, therefore (*by the 8.1.*) the angles A and D will be equal; and (*by the 26.*) the arches BC and EF will be also equal.

PROPOSITION XXIX.

A THEOREM.

The lines that subtend equal arches of equal circles are equal.

IF the lines BC and EF (*see fig. preced. Prop.*) subtend (or are the chords of) equal arches BC and EF in equal circles, they will be equal.

Demonstration.

The arches BC and EF are equal, and parts of equal circles; therefore (*by the 27.*) the angles A and D will be equal. Therefore in the triangles ABC, DEF, the sides AB, AC, DE, and DF being equal, as also the angles A and D; the bases BC, EF will be equal, (*by the 4.1.*)

The use.

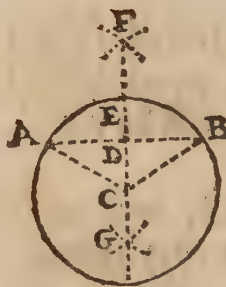
Theodorus by the 28 and 29 demonstrates, that

that the arches of the circles of the *Italian*
and *Babylonian* hours, contain'd between two
parallels, are equal. We have also demon-
strated after the same manner, that the arches
of the circles of the *Astronomical* hours, con-
tain'd between two lines parallel to the *Equino-*
tor, are likewise equal. These Propositions are
almost of continual use in spherical *Trigono-*
metry, and also in *Dialling*.

PROPOSITION XXX.

A PROBLEM.

To divide an arch of a circle into two equal
parts.



Suppose the arch AEB was to
be divided into two equal
parts. Place the foot of the
compass at the point A, and de-
scribe two arches F and G; then
removing it to the point B and
the same distance describe other
two arches, cutting the former in F and G;
the line GF will cut the arch AB equally at
the point E. Draw the line AB.

Demonstration.

By this operation you have divided the line AB
into two equal parts. For suppose there were
drawn

drawn the lines AF, BF; AG, and BG; (which have not done, least the figure should appear confus'd,) the triangles FGA and FGB would have all their sides equal, therefore (*by the 8. 1.*) the angles AFD, & BFD would be equal. Again, the triangles DFA and DFB have the side DF common, the sides AF and BF equal, and the angles DFA and DFB equal: therefore (*by the 1.*) the bases AD and BD are equal, and also the angles ADF and BDF. We have therefore divided the line AB equally and perpendicular-ly at the point D. Therefore (*by the 1.*) the center of the circle is in the line FG. Suppose it then to be the point C, and draw the lines CA and CB; all the sides of the triangles ACD and BCD are equal: therefore [*by the 8. 1.*] the angles ACD and BCD are equal, and [*by the 6.*] the arches AE and EB.

The USE.

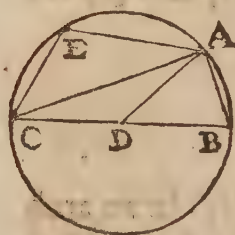
* Having frequent occasion to divide an arch into two equal parts, the exercise of this Proposition is very common. 'Tis thus that we divide the Mariners compass into 32 winds: for having drawn two diameters cutting each other at right angles, we divide the circle into four; and subdividing each quarter in the middle, we have eight, parts; and again subdividing those twice, we make 32. We have also

'also occasion for the same operation in the di-
 'viding a semicircle into 180 degrees; and
 'because to compleat that division we are obli-
 'g'd to divide an arch into three equal parts
 'all Geometricians have sought after a method
 'of doing that Geometrically, but have not yet
 'been so happy as to find one.

PROPOSITION XXXI.

A THEOREM.

*The angle in a semicircle is a right angle, that
 which is in a segment greater than a semicircle
 is an acute, and that which is in a lesser segment
 is an obtuse.*



IF the angle BAC be in a semi-
 circle, I will prove that it is
 a right angle. Draw the line
 DA. *Demonstration.*

The angle ADB being an ex-
 ternal angle in regard of the tri-
 angle DAC, is equal to both the internals DAC
 and DCA (*by the 32.1.*) and those being equal (*by*
the 5.1.) because the sides DA and DC are equal
 it will be double the angle DAC. In like man-
 ner the angle ADC is double the angle DAB
 therefore the two angles ADB, and ADC
 which are equal to two right angles, are double

the whole angle BAC, and consequently the angle BAC is a right angle.

Secondly, the angle AEC, which is in the segment AEC less than a semicircle, is an obtuse angle. For in the quadrilateral figure ABCE, the two opposite angles E and B are equal to two right angles, (*by the 22.*) but the angle B is an acute; therefore the angle E will be an obtuse.

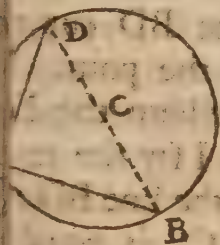
Thirdly, the angle B, which is in the segment ABC greater than a semicircle, is an obtuse; because in the triangle ABC, the angle ACB is a right angle.

The Use.

Mechanicks make use of this Proposition on to try if their Squares be just; for having describ'd a semicircle BAD, they lay down the point A of their square BAD upon the circumference, and one of its sides AB upon the point of the diameter B: and then the other branch AD ought to pass precisely to the point D, which is the other extreme of the diameter.

Ptolomey uses this Proposition to compose his table of Chords or Subtendants, of which he has occasion in his *Trigonometry*.

There is also a method of raising a perpendicular

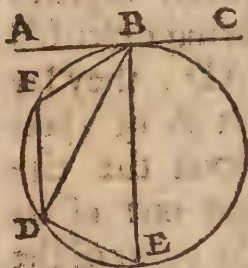


perpendicular at the end of a line, grounded upon
 this proposition. For example, to raise a perpendicular
 at the point A of the line AB. I place
 the foot of the compass upon the point C taken
 any where, and describe a circle through
 the point A, cutting the line AB at the point
 B. Then I draw the line BCD; and so it is
 evident, that the line AD is in a semicircle.

PROPOSITION XXXII.

A THEOREM.

*A line cutting a circle at the point of contact, makes
 with the tangent, angles, equal to those in the alternate
 segments.*



LET the line BD cut the
 circle at the point B,
 which is that where the line
 AB touches it. I say the angle
 CBD, made by the line BD and
 the tangent ABC, is equal to
 the angle F in the alternate
 segment BFD; and that the angle ABD is
 equal to the angle E in the segment BED.

First, if the line pass through the center,
 the line BE, it will make with the tangent two
 right angles, (by the 18.) and the angles of the
 semicircles would be also right angles, (by the
 pr

ceding,) therefore in this case the proposition would be true. But if the line do not pass through the center, as BD; draw the line BE through the center, and joyn the line DE.

Demonstration.

The line BE makes with the tangent two equal angles; and all the angles of the triangle BDE are equal to two right angles, (*by the 32. 1.*) therefore taking away the right angles BDE, and D which is in the semicircle, and likewise the angle EBD which is common to both, there will remain the angle ABD equal to the angle E.

Again, the angle CBD is equal to the angle E because in the quadrilateral figure BFDE, which is inscrib'd in a circle, the opposite angles E and F are equal to two right angles, (*the 22.*) but the angles ABD and CBD are equal to two right angles, [*by the 13. 1.*] and the angles ABD and E are equal, as I have now demonstrated: therefore the angles CBD and E are equal.

The Use.

This proposition is necessary to prove that which follows.

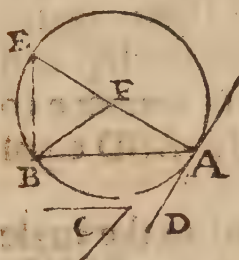
Z

PROP.

PROPOSITION XXXIII.

A PROBLEM.

Upon a line given to describe a segment of a circle capable of an angle given.



LET it be propos'd to describe a segment of a circle upon the line AB capable of the angle C. Make the angle BAD equal to the angle C, and draw AE perpendicular to AD, make also the angle ABF equal to the angle BAE: and in fine, from the point F, where BF and AE concur, at the distance BF or FA, describe a circle. The segment BEA is capable of an angle equal to the angle C.

Demonstration,

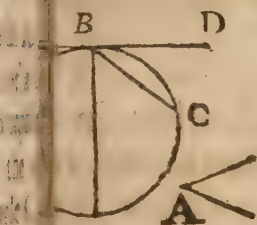
The angles BAF and ABF being equal, lines FA and FB are equal, [by the 6. 1.] and the circle, which is describ'd from the center by A, passes by B: Now the angle DAE being a right angle, the line DA touches the circle in A, [by the 16.] therefore the angle contained in the segment BEA, as the angle E, is equal to the angle DAB, that is the angle C, [by the preceding.] But if the angle given be an obtuse, we must take an acute, its complement to 90 degrees.

PRO

PROPOSITION XXXIV.

A PROBLEM.

circle being given, to cut a segment in it capable of a certain angle.



TO cut a segment of the circle BCE capable of the angle A, draw [by the 17.] the tangent BD, and make the angle DBC equal to the angle A. Tis evident [by the 32.]

that the segment BEC is capable of an angle equal to DBC, and consequently to the angle

The USE.

I have made use of this proposition to find geometrically the excentricity of the Annual Circle of the Sun, and his *Apogäum*, having three observations given. Tis used likewise in *Opticks*, to find a point where two unequal lines propos'd may appear equal, or under equal angles, by making upon each line segments which will contain equal angles.

Z z

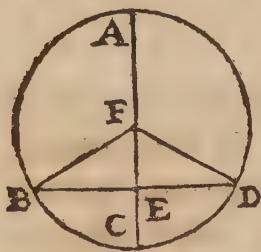
PROP.

PROPOSITION XXXV.

A THEOREM

If two lines cut each other within a circle, the rectangle contain'd under the parts of one is equal to the rectangle contain'd under the parts of the other.

First, if the two lines cut each other in the center, they will be both equal, and be equally divided; so that in that case it is evident, the rectangle contain'd under the parts of one, will be equal to the rectangle contain'd under the parts of the other.

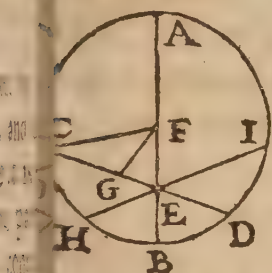


Secondly, if one of the lines pass through the center F, AC, and divide the line BD into two equal parts at the point E: I say, the rectangle contain'd under AE and EC, is equal to the rectangle contain'd under BE and ED, that is to say, to the square of BE. The line AC is perpendicular to BD, [by the 3.]

Demonstration.

Since the line AC is divided equally in F, and unequally in E, the rectangle contain'd under AE, and EC, with the square of FE, is equal

to the square of FC or FB, [*by the 5. 2.*] Now
the angle E being a right angle, the square of
FB is equal to the squares of BE and FE; there-
fore the rectangle under AE, EC, with the
square of FE, is equal to the squares of BE and
EF: and taking away the square of EF, there
remains the rectangle under AE, EC, equal to the
square of BE.



Thirdly, let the line pass
through the center F, and di-
vide the line CD into unequal
parts at the point E: draw FG
perpendicular to CD, and [*by
the 3.*] the lines CG and GD
will be equal.

Demonstration.

Since the line AB is divided equally in F
and unequally in E, the rectangle contain'd
under AE, EB, with the square of EF, is equal
to the square of FB, or FC, [*by the 5. 2.*] In-
stead of the square of EF put the squares of FG
and GE, which are equal to it, [*by the 47. 1.*]

In like manner the line CD being divided
equally in G, and unequally in E; the rectan-
gle under CE, ED, with the square of GE, is e-
qual to the square of GC. Add the square of GF;
the rectangle under CE, ED, with the squares
of GE and GF, will be equal to the squares of
CG and GF, that is to say, [*by the 47. 1.*] to
the square of FC. Therefore the rectangle un-
der

der AE, EB , with the squares of EG and GF , is equal to the rectangle under CE, ED , with the same squares : and consequently taking away the same squares from both, the rectangle AE, EB , will be equal to the rectangle CE, ED .

Fourthly, if the lines CD and HI , cut each other in E , neither of the two passing through the center: I say, the rectangle CE, ED is equal to the rectangle HE, EI . For drawing the line AFB , it is plain the rectangles CE, ED , and HE, EI , are both equal to the rectangle AE, EB , [*by the preceding case ;*] therefore they are equal betwixt themselves.

The USE.

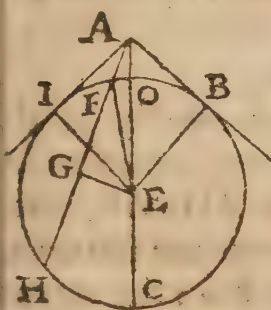
We are taught by this Proposition a method
 ' of finding a fourth proportional to three lines
 ' given, or a third proportional to two,

PROP.

A THEOREM.

Suppose the line AB to be drawn from the point A, taken without the circle, to touch the circle in B; and the line AC, or AH cutting it. The square of AB will be equal to the rectangle contain'd under AC, and AO, or the rectangle contain'd under AH, and the segment of the circle that pass through the center, as EB.

Since the line OC is divided in the middle at the point E, and the line AO added to it; the rectangle contain'd under AO and AC, with the square of OE or EB, will be equal to the square of AE, [*by the 6.2.*] Now the line AB is suppos'd to touch the circle at the point B: therefore [*by the 18.*] the angle B is a right angle,



gle, and (*by the 47. 1.*) the square of AE is equal to the squares of EB and AB; therefore the rectangle under AC and AO, with the square of EB, is equal to the squares of EB and AB: and taking away the square of EB from both, the rectangle under AC, AO will be equal to the square of AB.

Secondly, suppose the secant AH not to pass through the center; and draw the line EG perpendicular to EH, which will divide in the middle the line FH at the point G; draw also the line EF.

Demonstration.

The line FH being divided equally at the point G, and the line AF being added to it; the rectangle contain'd under AH, AF, with the square of FG, will be equal to the square of AG. Add to both the square of EG: the rectangle under AH, AF, with the squares of FG and GE, that is (*by the 47. 1.*) the square of FE, or EB, will be equal to the squares of AG and GE, that is, (*by the 47. 1.*) the square of AE. Further, the square of AE (*by the same*) is equal to the squares of EB and AB: therefore the rectangle contain'd under AH, AF, with the square of BE, is equal to the squares BE and AB: and taking away the square of BE from both, the rectangle contain'd under AH, AF will be equal to the square of AB.

Coroll. 1. If you draw divers secants from the same

the point, as AC and AH, the rectangles under AC and AO, AH and AF, will be equal twixt themselves, since they are both equal to the square of AB.

Coroll. 2. If you draw two tangents from the same point, as AB, AI, they will be equal: because the squares will be equal to the same rectangle under AC, and AO, and consequently twixt themselves; as also the lines.

PROPOSITION XXXVII.

A THEOREM.

the rectangle contain'd under the secant and the external line be equal to the square of a line that falls upon the circle, that line will touch the circle.

Suppose the secant to be AC or AH, and the rectangle AC, AO; or AH, AF, (see fig. pre-
d.) to be equal to the square of the line AB; the line AB will touch the circle. Draw the tangent AI, (by the 17.) and the line IE.

Demonstration.

Since the line AI touches the circle, the rectangle AC, AO; or AH, AF, will be equal to the square of AI. But the square of AB is suppos'd to be equal to either of those rectangles; therefore the squares of AI and AB are equal, and consequently the lines AI and AB. Therefore

the triangles ABE and AIE, having all sides equal, will be equiangular, (by the 8. 1.) and because the angle AIE is a right angle (by the 18.) the line AI being a tangent, the angle ABE will be a right angle, and the line AB tangent, (by the 16.)

The Use.

‘*Magnylocus* makes use of this Proposition to find the diameter of the Earth. For observing from the top of a mountain OA, the superficies of the earth by the line BA, he takes notice of the angle OAB, made by the line AB and a perpendicular AC: and by *Trigonometry* calculates the length of the line AB. Then multiplying AB by AB to have its square, he divides the product by AO the height of the mountain, which gives the quotient AC, the diameter of the earth, with the height of the mountain; from which having subtracted AO, there will remain OC the diameter of the earth. This proposition serves also to prove the fifth of the third book of *Trigonometry*.

THE FOURTH BOOK

OF THE

ELEMENTS
OF
EUCLID.

THIS fourth book is exceeding useful in *Trigonometry*. For by inscribing Polygons in a Circle, we learn the methods of composing the Table of Subtendants, Tangents, and Secants; a practice most necessary for taking all sorts of Dimensions.

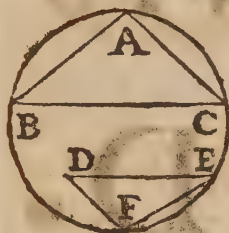
Again, by inscribing Polygons in a circle, we find the divers Aspects of the Stars, which also take their names from those Polygons.

Thirdly, the same Operations give us the *Quadrature* of the Circle, as exact as is needful. And by them we also demonstrate, that Circles are in the duplicate proportion of that of their Diameters.

Fourthly, Military Architecture does frequently

‘quently require to inscribe Polygons in a circle,
 ‘cle, to compose the designs and platforms of
 ‘regular Fortifications.

DEFINITIONS.



1. **A** Rectilineal figure is inscribed in a circle, or a circle is describ'd about it, when all its angles are in the circumference of the same circle.

‘As the triangle ABC is inscribed in a circle, and the circle is describ'd about the triangle; because its angles A, B, and C, do all terminate at the circumference. The triangle DEF is not inscribed in the circle, because the angle D does not terminate at the circumference of the circle.

2. A rectilineal figure is describ'd about a circle, and the circle inscrib'd within that figure, when all the sides of the

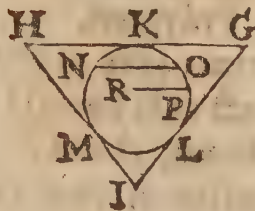


figure touch the circumference of the circle. “ as the triangle GHI is describ'd about the circle KLM because its sides touch the circumference of the circle in K, L, and M.

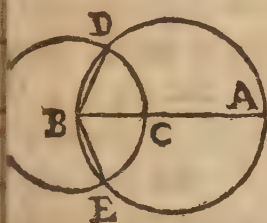
3. A line is apply'd to, or inscrib'd in a circle,

le, when its two extrems touch the circumference of the circle. " As the line NO. But
" the line RP is not inscrib'd in the circle.

PROPOSITION I.

A PROBLEM.

To inscribe in a circle a line, that does not exceed its diameter.



Let a line be propos'd to be inscrib'd in the circle AEBD, not exceeding its diameter. Take the length of the line propos'd upon the diameter ; for example, let it be BC. Place the foot of the compass upon the point B, and describe a circle at the distance of BC, which may cut the circle AEBD in D and E. Then draw the line BD or BE. Tis evident they are equal to BC, (*by the definition of a circle.*)

The USE.

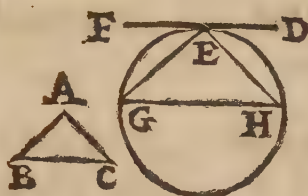
" This proposition is necessary for the performance of what is requir'd in the following.

PROP.

PROPOSITION II.

A PROBLEM.

To inscribe in a circle a triangle equiangular to any other triangle.



LET the circle be EGH in which a triangle is to be inscrib'd, equiangular to the triangle ABC. Draw the tangent FED , (by the 17. 3.) and at the point of contact E make the angle DEH equal to the angle B , and the angle FEG equal to the angle C , [by the 23. 1.] and draw the line GH ; the triangle EGH will be equiangular to the triangle ABC .

Demonstration.

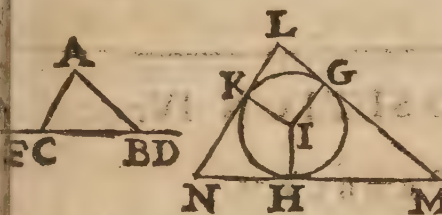
The angle DEH is equal to the angle EGH of the alternate segment, [by the 32. 3.] Because the angle DEH is equal to the angle B , and consequently the angles B and G are equal. By the same reason the angles C and H are also equal and [by Coroll. 2. of the 32. 1.] the angles A and GEH will be equal. Therefore the triangle EGH and ABC are equiangular.

PROI

PROPOSITION III.

A PROBLEM.

To describe a triangle about a circle equiangular to another triangle.



IF you would describe a triangle equiangular to the triangle ABC about the circle GKH.

Continue one of the sides of the triangle given C to D and F, and make the angle GIH equal to the angle ABD, and HIK equal to the angle BCF: then draw the tangents LGM, LKN, and NHM, through the points G, K, and H. These tangents will concur; because the angles KGL and IGL being right angles, if you should draw a line KG, the angles KGL and GKL would be less than two right angles: therefore the lines GL and KL must concur, [by the 11. Axiom.]

Demonstration.

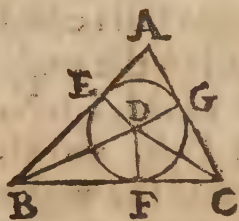
All the angles of the quadrilateral GIHM are equal to four right angles, because it may be divided into two triangles. The angles IGM and IHM, which are made by the tangents, are

are right angles ; therefore the angles M and N are equal to two right angles, as are also the angles ABC and ABD. But the angle GIH is equal to the angle ABD, therefore the angle M will be equal to the angle ABC. By the same reason the angles N and ACB are equal, and therefore the triangles LMN and ABC are equiangular.

PROPOSITION IV.

A PROBLEM.

To inscribe a circle in a triangle.



IF you would inscribe a circle in the triangle ABC, divide the angles ABC and ACB into two equal parts, [by the 9. 1.] drawing the lines BD and CD, which will concur at the point D. This done, from the point D draw the perpendiculars DE, DF, and DG, which will be equal ; so that a circle described from the center D, at the distance DE, will pass through F and G.

Demonstration.

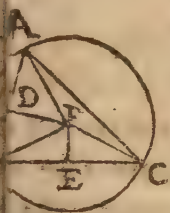
The triangles DEB and DFB have the angle DEB and DFB equal, being both right angles ; the angles DBE and DBF are also equal, therefore the angles

the ABC having been divided into two equal
 parts; and the side DB is common: therefore
 (the 26.1.) the triangles will be equal in all
 respects, and the sides DE and DF will be equal.
 In the same manner might I demonstrate
 the sides DF and DG to be equal. 'Tis possible
 therefore to describe a circle, which shall pass
 through the points E, F, and G; and because
 the angles E, F, and G are right angles, the sides
 AC, and BC will touch the circle, which
 consequence is inscrib'd in the triangle.

PROPOSITION V.

A PROBLEM.

To describe a circle about a triangle.



IF you would describe a circle
 about the triangle ABC, di-
 vide the sides AB and BC into
 two equal parts, at the points D
 and E, drawing the perpendicu-
 DF and EF, which will concur at the point
 F. Which done, if you describe a circle from
 center F, at the distance FB, it will pass
 through A and C; that is to say, the lines FA,
 FB, and FC, are equal.

Demonstration.

The triangles ADF and BDF have the side
 AD B b DF

DF common, and the sides AD and DB equal, the side AB having been divided equally in H , and the angles at D are equal, being right angles. Therefore (by the 4. 1.) the bases AF and BF are equal; as also the bases BF and CF .

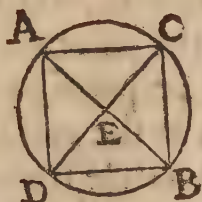
The U S E.

' I have frequent occasion to inscribe a triangle in a circle; as, for instance, in the first proposition of my 3. book of Trigonometry. This performance also is necessary in the measuring the *area* of a triangle; and in many other occasions.

PROPOSITION VI.

A PROBLEM.

To inscribe a square in a circle.



TO inscribe a square in the circle $ACBD$, draw the diameter AB , and perpendicular to it the line DC passing through the center E ; then draw the lines AC , CB , BD , DA , and you will have inscrib'd in the circle the square $ACBD$.

Demonstration.

The triangles AEC and CEB have their sides equal, and the angles AEC and CEB equal, being

ing both right angles : therefore their bases AC and CB are equal, (*by the 4. 1.*) Further, because the sides AE and EC are equal, the angles EAC and ECA will be equal : and the angle E being a right angle, they will be half-right angles, (*by the 32. 1.*) therefore the angle ACB is half a right angle, and consequently the angle ACB will be a right angle. And the same reason holds for all the rest : therefore the figure ACBD is a square.

PROPOSITION VII.

A PROBLEM.

To describe a square about a circle.



HAVING drawn the two diameters AB, and CD, which cut each other perpendicularly at the center E, draw the tangents FG, GH, HI, and IF, by the points A, B, C, and you will have describ'd the square GHFI, about the circle ACBD.

Demonstration.

The angles E and A are right angles, therefore (*by the 27. 1.*) the lines FG and CD are parallels. After the same manner I may prove, that CD and HI, FI and AB, AB and GH, are parallels. Therefore the figure FCDG is a

B b 2

paralle-

parallelogram, and (*by the 34. 1.*) the lines FI and CD are equal, as also the lines CD and FI ; and consequently the sides of the figure FG and HI are equal. Further, since the lines FG and CD are parallel and the angle CDG is a right angle, the angle FGH will also be a right angle, (*by the 29. 1.*) In the same manner I may demonstrate the angles F , H , and I , to be right angles. Therefore the figure $FGHI$ is a square, whose sides touch the circle.

PROPOSITION VIII,

A PROBLEM.

To inscribe a circle in a square,

IF you would inscribe a circle in the square $FGHI$, [*see fig. preced.*] divide the sides FG , GH , HI , IF , in the middles at the points A , D , B , C , and draw the lines AB and CD which may cut each other at the point E . I demonstrate that the lines EA , ED , EB , and EC are equal, and the angles A , D , B , C , right angles: and that therefore you may describe a circle from the center E , which will pass through A , D , B , and C , and touch the sides of the square $FGHI$.

Demo

Demonstration.

Since the lines AB and GH do conjoyn the lines AG and BH, which are parallel and equal, they also will be parallel and equal: therefore the figure AGDE is a parallelogram; and the lines AE and GD, AG and ED are equal: and AG and GD being equal, AE and ED will be equal also. Tis after the same manner that the lines AE, EC, EB, are prov'd equal. Further, AG and CD being parallel, and the angle G a right angle, the angle D will be so likewise. Therefore the circle ADBC may be describ'd from the center E, which will pass through the points A, D, B, C, and touch the sides of the square.

PROPOSITION IX.

A PROBLEM.

To describe a circle about a square.

TO describe a circle about the square ACB D, [see fig. in Prop. 6.] draw the diagonals AB, and CD, which will cut each other at the point E; the point E will be the center of the circle, which will pass through the points A, C, B, D. It ought therefore to be demonstrated, that the lines AE, EB, CE, and ED are equal.

Demon-

Demonstration

The sides AC and CB are equal, and the angle C is a right angle; therefore the angles BAC and ABC are equal, (*by the 5. 1.*) and half right angles, (*by the 32. 1.*)

After the same manner I demonstrate, that the angles ACD, ADC, BDC, and BCD, are half right angles. Therefore the triangle AEC having the angles EAC and ECA half right angles, and consequently equal, will have also (*by the 6. 1.*) its sides AE and EC equal. The same may be prov'd of the lines EC and EB, EB and ED, that they likewise are equal.

The USE.

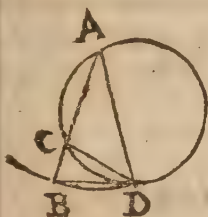
' We show in the 12. book, that Polygons inscrib'd in a circle, degenerate into circles
' and as these Polygons are always in the duplicate proportion of that of their diameters, so
' likewise are circles. In practical Geometry
' we have frequent occasion to inscribe a square
' and other Polygons in a circle, or to describe
' them about it, to reduce a circle to a square.

PROF

PROPOSITION X.

A PROBLEM.

To describe an Iſosceles (or equicrural triangle) having its angles at the baſe, each of them double to the third angle.



TO describe an *Isoceles* ABD, having each of its angles ABD and ADB, double the angle A, divide the line AB (by the 11.2.) so that the square of AC may be equal to the rectangle under AB and BC; and from the center A at the distance AB describe the circle BD, in which inscribe BD equal to AC; and drawing the line DC describe a circle about the triangle ACD, (by the 5.)

Demonstration.

Since the square of AC or BD is equal to the rectangle contain'd under AB and BC, the line BD will touch the circle ACD at the point D, (*by the 37.3.*) therefore the angle BDC will be equal to the angle A, being in the alternate segment CAD, (*by the 32.3.*) Now the angle BCD, being an external angle in respect of the triangle ACD, is equal to the angles A and CDA:

CDA;

CDA ; therefore the angle BCD is equal to the angle BDA. Further, the angle ADB is equal to the angle ABD, (*by the 5.1.*) therefore DCB and DBC are equal, and (*by the 6.1.*) the sides BD and DC will be equal : and since BD is equal to AC, the sides AC and CD will be equal ; and so likewise the angles A and CDA. Therefore the angle ADB is double the angle A.

PROPOSITION XI.

A PROBLEM.

To inscribe a regular Pentagon in a circle.



TO inscribe a Regular Pentagon in a circle, describe (*by the 10.*) an *Isosceles* ABC, having each of its angles ABC, ACB, at the base, double the angle

A. Inscribe in the circle the triangle DEF equiangular to ABC : then divide the angles DEF and DFE into two equal parts, drawing the lines EG and FH. Lastly, joyn the lines DH, DG, GF, EH, and you will have describ'd a regular Pentagon ; that is to say, a Pentagon having equal sides, and equal angles.

Demon.

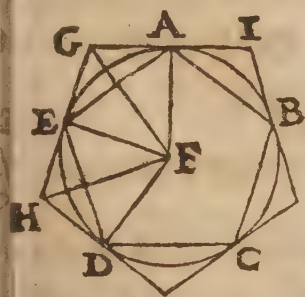
Demonstration.

The angles DEG, GEF, DFH, and HFE, being the halves of the angles DEF and DFE, each of which is double to the angle EDF, are equal to the angle EDF: and consequently the five arches, which are their bases, are equal, (*by the 26.3.*) and the lines DH, HE, EF, FG, and GD, are equal, (*by the 29.3.*) Secondly, the angles DGF, GFE, and so of the rest, having each three of those equal arches for its base, will be also equal, (*by the 27.3.*) Therefore the sides and angles of the Pentagon DHEFG are equal.

PROPOSITION XII.

A PROBLEM.

To describe a Pentagon about a Circle.



INScribe a regular Pentagon ABCDE in the circle, (*by the 11.*) and having drawn tangents through the points A, B, C, D, E, (*by the 17.3.*) you will have describ'd a regular penta-

gon about the circle. Draw the lines FA, FG, FE, FH, FD.

Demonstration.

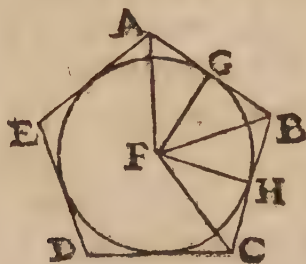
The tangents GE and GA are equal, (*by co-
C c* *rol.*

roll. 2 of the 36. 3.) as also EH, and HD : the lines FA and FE are also equal (*by the definit. of a circle*;) therefore (*by the 8. 1.*) the triangles FGA and FGE are equal in all respects; and the angles AFG and EFG are equal, as also the angles EFH and DFH. And because (*by the 27. 3.*) the angles EFA and EFD are equal, their halves EFH and EFG will be equal; and (*by the 26. 1.*) the triangles EFH and EFG will be equal in all respects, and the sides EG and EH will be also equal. After the same manner I can demonstrate all the sides to be divided into two equal parts; and consequently, since the lines GE and GA are equal, GH and GA will be also equal. Further, the angles G and H being double the angles FGE, and FHE are also equal, Therefore we have described a regular Pentagon about the circle.

PROPOSITION XIII.

A PROBLEM.

To inscribe a circle in a regular Pentagon.



TO inscribe a circle in the regular pentagon ABCDE, divide the angles A and B into two equal parts by the lines AF and BF, which concur at the point F.

Point F. Then drawing the line FG perpendicular to AB, describe a circle from the center F at the distance FG. I say it will touch all the other sides; that is to say, having drawn FH perpendicular to BC, FH and FG will be equal.

Demonstration.

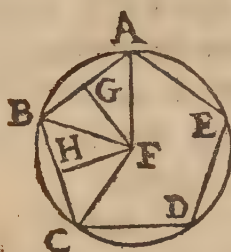
Since the equal angles A and B were divided into two equal parts, their halves GAF and FBF will be equal: and since the angles at G and H are right angles, the triangles AFG and BFG will be equal in all respects, (by the 26.1.) therefore the lines AG and GB are equal.

Further, I may prove the lines BG and BH, also the lines FG and FH, to be equal; and the sides AB and BC of a regular pentagon being equal, the lines BH and HC will be equal; and consequently, the angles at the point H being also right angles, and equal, the triangles FGH and FHC will be equal in all respects, and the angles of FBH and FCH will be equal. And since the angles B and C are equal, the angle FCH will be half the angle C. So passing from one to the other I will demonstrate, that the perpendiculars FG, and FH, and the rest, are equal.

PROPOSITION XIV.

A PROBLEM.

To describe a circle about a regular Pentagon



TO describe a circle about a regular pentagon ABCDE, divide equally two of its sides AB and BC at G and H, and draw the perpendiculars CG and HF. The circle drawn from the center F, at the distance FA, will pass through B, C, D, E.

Demonstration.

Suppose the circle describ'd, it is evident (by the 1. 3.) that having divided the line AB in the middle in G, and drawn the perpendicular GF, the center of the circle must be in that perpendicular: it is also in HF: therefore it is at the point F.

The USE.

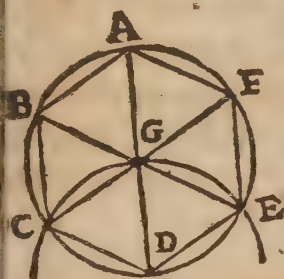
' These Propositions are solely useful for
' composing the table of Sines, and drawing
' platforms of Cittadels, for their ordinary
' figures are pentagons. Observe also that the
' methods of describing pentagons about a
' circle, may be apply'd likewise to other P

gons. But in my book of Military Architecture, I have shown another way of inscribing a regular Pentagon in a circle.

PROPOSITION XV.

A PROBLEM.

To inscribe a regular Hexagon in a Circle.



TO inscribe a regular Hexagon in the circle ABC DEF, draw the diameter AD, and fixing the foot of the compass at the point D describe a circle at the distance of DG: then draw the dia-

meters EGB, and CGF; and the lines AB, AF, FE, and the rest.

Demonstration.

Tis evident, that the triangles CDG, and DGE, are equilateral; therefore the angles CGD, DGE, and those oppos'd to them at the top, BGA, and AGF, are each of them the third part of two right angles; that is to say, contain 60 degrees. Now all the angles that can be made about the same point are equal to four right angles, that is to say, 360 degrees.

Therefore taking away four times 60, that is 240, from 360, there will remain 120 for BGC and FGE; which therefore each contain

60

60 degrees. Therefore all the angles at the center being equal, all the arches and all the sides will be equal; and every angle as A, B, C, &c. will be compounded of two angles of 60 degrees each, that is, 120 degrees, and therefore will be equal.

Coroll. The side of a Hexagon is equal to the semidiameter.

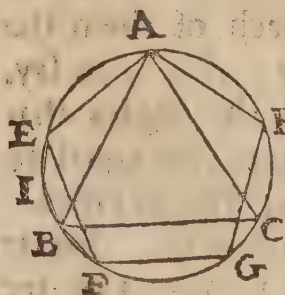
The USE.

'Because the side of a Hexagon is the base of an arch of 60 degrees, and is equal to the semidiameter, its half will be the sine of 30; by which sine we begin the table of Sines. *Euclid* treats of Hexagons in the last book of his *Elements*.

PROPOSITION XVI.

A PROBLEM.

To inscribe a regular Pentedecagon in a circle.



INscribe in the circle an equilateral triangle ABC (*by the 2.*) and a regular pentagon (*by the 11.*) so that the angles may meet at the point A. The lines BF, BI, and IE, will be the sides of a Pentedecagon: and if you inscribe in the other arches

Arch lines equal to BF and BI , you will complete the Polygon.

Demonstration.

Since the line AB is the side of an equilateral triangle, the arch AEB will be the third part of the whole circle, that is, five fifteenths, but the arch AE being the fifth part, will contain three fifteenths; therefore the arch EB contains two: and if you divide it in the middle at the point I , each part will be a fifteenth.

The Use.

‘ This Proposition serves only to open the way to other Polygons. We have in the *Compass of Proportion* some most easie methods of inscribing all ordinary Polygons, but they are grounded on this here. For it would be impossible to mark polygons upon that instrument, if their sides were not first found by this, or other like propositions.

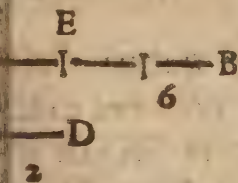
THE

THE FIFTH BOOK
OF THE
ELEMENTS
OF
EUCLID.

THis fifth Book is absolutely necessary
to demonstrate the propositions of the sixth
Book. The doctrine it contains is of universal use
and its manner of argumentation by Propo-
sition the most subtil, solid, and brief. In-
deed much that all those Treatises, that are ground-
ed on Proportions, are oblig'd to make use
thereof, as a kind of *Mathematical Logic*
Geometry, *Arithmetick*, *Musick*, *Astronomy*,
Staticks, and in a word all parts of *Mathem-*
atics, are constrain'd to borrow some of their
demonstrations from the Propositions of this
Book. The greatest part of Measurings

Practical Geometry is done by Proportion. All the rules of *Arithmetick* are demonstrable by the Theorems that occur here; so that there is no necessity of having recourse to the seventh, eighth, or ninth books for that purpose. The *Musick* of the Antients is scarce any thing else but the doctrine of Proportions apply'd to Sounds. The same may be said of *staticks*, which considers the proportion of Weights. In fine tis most certain, that if the knowledge of Proportions, which this book affords, was taken away from the *Mathematicks*, what remain'd would be very inconsiderable.

DEFINITIONS.



A Small quantity compar'd with a greater, is call'd a Part. " As for example, if the line CD of two foot long be compar'd with AB of 6, it will be call'd its part. Which name also it obtains, though indeed it be not contain'd in AB, provided AE, a line equal to CD, be therein found. " The whole is answerable to the part, and therefore will be the greater quantity compar'd

'par'd with the less; whether it do really con-
'tain the less, or not.

'A Part taken in general is ordinarily divid-
'into (that which is call'd) an *Aliquot* part, and
'an *Aliquant* part.

1. An *Aliquot* part (which alone *Euclid* defines in this book) is a magnitude of a magnitude, a less of a greater, when it exactly measures the greater. "That is to say, 'Tis a less quantity compar'd with a greater, which precisely measures the greater. As a line two feet long taken three times, is equal to a line six feet in length.

2. A *Multiple* is a magnitude of a magnitude, a greater of a less, when it is exactly measured by the less. "That is to say, A *Multiple* is a greater quantity compar'd with a less, which it contains exactly so many times. For example, a line six foot long is triple a line two foot long.

An *Aliquant* part, is a lesser quantity compar'd with a greater, which it does not exactly measure. As a line of four feet in length is an aliquant part of a line ten foot long. "In other word, An *Aliquot* part so many times repeated will equal the whole: but an *Aliquant* part though it contains such a quantity of the whole, yet repeated as you please will not be exactly equal, but either come short of, or exceed, the whole.

4; 6, 2.
B; C, D

Equimultiples are magnitudes which equally contain their aliquot parts. "That is to say, so many times. As for example, if A contains B many times as C contains D, A and C will be equimultiples of B and D.

3. * *Proportion* is a respect of one magnitude another of the same kind.

* *Græc. ἀνύσθησις. Gall. Raïson.*

4. *Quantities* are said to have a certain proportion to each other, when being multiply'd they can exceed each other. "For which reason they ought to be of the same kind. For indeed a line has no proportion to a superficies; because a line taken *Mathematically* is consider'd without any breadth at all; and therefore multiply'd as much as you please will never give any breadth, which yet a superficies contains.

'For as much as Proportion is a respect or relation founded upon quantity, it ought to have two terms. That which some Philosophers would call the *Fundamentum*, or Foundation, *Mathematicians* name the *Antecedent*, and the *Term* is call'd by them the *Consequent*. As if we were to compare the quantity A with the quantity B, that respect or proportion would have for the Antecedent the quantity A, and for the consequent the quantity B.

Dd 2

On

'On the contrary, if B be compar'd with
'that proportion of B to A would have B
'the antecedent, and A for the consequent.

'Proportion, or the respect of one quantity
'to another, is divided into Rational Proportion,
'and Irrational.

Rational Proportion is the respect of one
'quantity to another, which is commensurable
'to it, that is, when both the quantities have
'some common measure, by which both may
'exactly measur'd. As the proportion of a line
'four foot long to another that is six, is rational
'because a line two foot long may exactly measure
'both. And when this happens, these quantities
'have the same proportion as one number
'to another. For example, since the line two
'foot long, which is their common measure,
'is found twice in the four-foot line, and three
'times in that which is six foot long; the first has
'the same proportion to the second, as 2 to 3.

'Irrational Proportion is betwixt two quantities
'of the same kind, which are incommensurable,
'i. e. have no common measure. As the
'proportion of the side of a square to its diagonal.
'For there is no measure so small, as will
'precisely measure both.

'Four quantities will be proportionals, when
'the proportion of the first to the second, is
'the same with, or like to, that of the third to
'the fourth; so that, to speak properly, Proportions

onality is a similitude of proportions. But it is
no easie matter to understand in what consist
this similitude of proportions; that is to say,
how two respects or relations may be alike.
For *Euclid* has not given us a just definition
thereof, or such an one as might explain the
nature of the thing, but contented himself to
set down some marks or signs, by which it
may be known, whether or no quantities had
the same proportion. And tis the obscurity of
this definition, which has render'd the whole
book so difficult to be understood; which de-
fect therefore I shall endeavour to supply.

5. *Euclid* makes four magnitudes to have the
same proportion, when taking the Equimulti-
ples of the first and the third, and likewise the
Equimultiples of the second and the fourth,
according to any multiplication whatsoever:
if the multiple of the first exceed that of the
second, the multiple of the third will also ex-
ceed that of the fourth; and if it be equal to,
or less than the second, the third will be equal
to or less than the fourth. In such a case the
first has the same proportion to the second, as
the third to the fourth.

As

A, B :: C, D

2, 4 :: 3, 6.

E, F :: G, H

10, 8 :: 15, 12

K, L :: M, N

8, 8 :: 12, 12

O, P :: Q, R

6, 16 :: 9, 24

‘As for example, if four
‘magnitudes were propos’d,
‘A, B, C, D; having taken
‘the Equimultiples of A
‘and C, as their quintuples
‘E and G; and F and H the
‘doubles of B and D; In
‘like manner taking K and
‘M the quadruples of A and
‘C, and L and N the doubles of B and D; A-
‘gain taking O and Q the triples of A and C,
‘and P and R the quadruples of B and D; Be-
‘cause E being greater than F, G is greater
‘than H; and K being equal to L, M is equal
‘to N; and lastly, O being less than P, Q is
‘less than R: therefore A will have the same
‘proportion to B, as C to D. But methinks *Eu-*
‘clid ought to have demonstrated this proposi-
‘tion, it being too perplex’d and obscure to pass
‘for a principle.

‘To explain aright what Proportionality is,
‘or how four magnitudes may be in the same
‘proportion; though it may be sufficient to say
‘in general, that the first ought to be a like
‘part, or a like whole in respect of the second;
‘as the third is, compar’d with the fourth: yet
‘because this definition agrees not to the pro-
‘portion of Equality, I shall give a more gene-
‘ral one; and to make it the more intelligible
‘explain first what is a similar or like Aliquot
‘part.

Similar

'Similar Aliquot parts, then are such as are
'contain'd in their wholes as many times one
'as the other : as 3 in respect of 9, and 2 in re-
'spect of 6, are similar aliquot parts; because they
'are each contain'd three times in their respec-
'tive wholes.

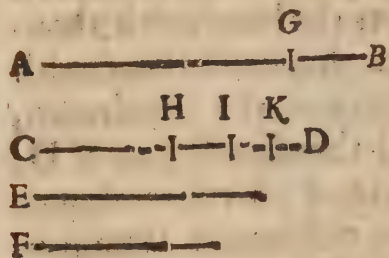
'The first quantity will have the same propor-
'tion to the second, as the third to the fourth,
'if the first contains so many times such aliquot
'parts of the second, as the third contains the
A, B, C, D, | 'like aliquot parts of the fourth. As
'if A contains the hundredth, thou-
'sandth, or hundred-thousandth
'part of B, as oft as C contains the hundredth,
'thousandth, or hundred-thousandth part of D;
'(and the like may be said of all other aliquot
'parts imaginable :) there will be the same pro-
'portion of A to B, as of C to D.

To render this Definition still more clear, I
'will prove first that, if A has the same propor-
'tion to B as C to D, A will contain the ali-
'quot parts of B, as oft as C does the like of D;
'and secondly, if A contain the aliquot parts of
'B as oft as C does the like of D, then there
'will be the same proportion of A to B, as of
'C to D.

'The first point seems sufficiently evident
'from the very notion of the terms: for if A
'contains the tenth part of B once more than an
'hundred times, and C contains the tenth part
of

'of D an hundred times only ; the quantity A
'will be a greater Whole compar'd to B, than
'C compar'd to D ; therefore it cannot be com-
'par'd after the same manner, the respect or re-
'lation being not the same.

'The second point seems more difficult, *viz.*
if a quantity, suppose AB, contain the aliquot
'parts of another, CD, as oft as a third, E, con-
'tains the like of a fourth, F : there will be
'the same proportion of AB to CD, as of E to



'F. But if it be other-
'wise, let us suppose
AB to have a greater
proportion to CD
'than E has to F ; that
'is to say, that AB is
'too great to have the

'same proportion to

CD, that E has to F. Therefore a quantity less
'than AB, as AG, will have the same propor-
'tion to CD, as E to F. Divide therefore CD in
'the middle in H, and HD in the middle in I
'and ID in the middle in K ; continuing the
'like division till you arrive at an aliquot part
'of CD less than GB, which I will suppose to
be KD.

Demonstration.

'Since there is the same proportion of A
'to CD, as of E to F ; AG will contain K
'an aliquot part of CD, as oft as E contains the

like aliquot part of F. Now AB will contain KD once more than E contains the like aliquot part of F; which is contrary to the supposition.

6. The first Quantity is said to have a greater proportion to the second, than the third to the fourth, when the first contains a certain aliquot part of the second oftner than the third contains the like aliquot part of the fourth: As 101 has a greater proportion to 10 than 200 to 20, because 101 contains the tenth part of 10, that is, 1, once above a hundred times; and 200 contains the tenth part of 20, i.e. 2, a hundred times only.

7. Magnitudes or Quantities having the same proportion, are called Proportionals.

8. * Proportionality, or Analogy, is a similitude of Proportions or Respects.

* *Ἀναλογία. Eucl.*

9. In each proportionality are required at the least three terms. "For that there may be a similitude of proportions, there must be two of them: and every proportion having two terms, an Antecedent and a Consequent, there seems to be a necessity of four terms; as when we say, that A has the same proportion to B as C to D; but because the consequent of the first proportion may be the antecedent of the second, three terms may suffice; as when A is said to have the same proportion to B as B to C.

E c

10. Mag-

10. Magnitudes are said to be continually proportionals, when the intermediate terms are taken twice, *i. e.* both as antecedents and consequents. "As if there be the same proportion of A to B, as of B to C, and of C to D,

11. In that case A will have the duplicate proportion to C, and the triplicate to D, of what it has to B.

'But here it is to be observ'd, that there is a great deal of difference between double proportion, and duplicate. We say that the proportion of four to two is double, because four is the double of two; the number two giving the name to the proportion, or rather to the antecedent thereof. Accordingly double, triplicate, quadruple, quintuple, &c. are denominations drawn from the numbers two, three, four, five, &c. compar'd with unity; which I insert in, because we more easily conceive the proportion, the less are its terms. But, as I said, these denominations do rather affect the antecedents, than the proportions themselves; for we call that double or triple proportion whose antecedent is double or triple its consequent. But by duplicate proportion we understand such an one, as is compounded of two similar proportions. As, if there be the same proportion of two to four, as of four to eight, the proportion of two to eight being compounded of the proportion of two to four, and

that of four to eight, which are similar and equal, will be the duplicate of each of them. So three to twenty seven is the duplicate proportion of that of three to nine. The proportion of two to four is call'd the sub-double, because two is the half of four; but that of two to eight is the duplicate of the sub-double; which is as much as to say, that two is the half of half of eight, as three is the third part of the third part of twenty seven; where you may observe, that the Denominators $\frac{1}{2}$ and $\frac{1}{2}$ be taken twice.

In like manner the proportion of eight to two is a duplicate proportion of that of eight to four, because eight is the double of four, but it is the double of the double of two. If there be four terms in continual proportion, the proportion of the first to the last is a triplicate of that of the first to the second; as in these four numbers, Two, Four, Eight, Sixteen; the proportion of two to sixteen is a triplicate of that of two to four, because two is the half of the half of the half of sixteen. So also the proportion of sixteen to two is a triplicate of that of sixteen to eight, because sixteen is the double of the double of the double of two.

12. Antecedents to antecedents, and consequents to consequents, are call'd Homologous magnitudes. As if there be the same proportion

Ec 2

'of A to B, as of C to D; A and C, B and D
'are homologous.

'The following Definitions explain the diverse
'manners of arguing from proportionals: for the
'demonstration of which this book was principally
'pally compos'd.

13. Alternate proportion is when we compare the antecedent of the one with the antecedent of the other, and the consequent of the one with the consequent of the other. "As for example, if because there is the same proportion of A to B, as of C to D, I infer, that there is the same proportion of A to C, as of B to D. This manner of argumentation holds only when all the four terms are of the same species or kind; i.e. either all lines, or all superficieses or all solids. This is demonstrated *Prop. 16.*

14.*Inverted Proportion is the comparing the consequents with the antecedents.

Ἀντιπαλιν *Eucl. Converse. Gall.*

'As, if because there is the same proportion of A to B, as of C to D, I conclude that there is the same proportion of B to A, as of D to C. *Corol. of Propos. 16.*

15. Composition of Proportion is the comparing of the antecedent and the consequent taken together, with the consequent alone. "As if, because there is the same proportion of A to B, as of C to D, I conclude that there is the same proportion of A and B to B, as of C and D to D. *Prop. 18.*

16. I

16. Division of Proportion is the comparing of the Excess of the antecedent above the consequent to the same consequent. "As, if there be the same proportion of AB to B, as of CD to D ; from thence I infer, that there is the same proportion of A to B, as of C to D. *Proposition 17.*

17. Conversion of Proportion is the comparing of the antecedent with the Difference of the Terms : "As, if there be the same proportion of AB to B, as of CD to D. I thence conclude, that there is the same proportion of AB to A, as of CD to C. *Coroll. of Prop. 18.*

18. Proportion of Equality is the comparing of the extreme magnitudes, and omitting those in the middle. "As if, there being the same proportion of A to B, as of E to F ; and of B to C, as of F to G ; and of C to D, as of G to H ; I infer, that there is the same proportion of A to D, as of E to H.

19. Proportionality of Equality orderly plac'd, is that in which the terms are compar'd together in the same order. "As in the foregoing example. *Prop. 22.*

20. Proportionality of Equality disorderly plac'd, is that in which the terms are compar'd in a different order. "As if, there being the same proportion of A to B, as of G to H ; and of B to C, as of F to G ; and of C to D, as of E

'E to F ; I conclude that there is the same proportion of A to D, as of E to H. *Prop. 23.*

'See in short all the different manners of argumentation by Proportion.

'As A to B, so C to D ; therefore

'By Alternate proportion : as A to C, so B to D :

'Inverted : As B to A, so D to C :

'Composition. As AB to B, so CD to D :

'Division. If as AB to B, so CD to D : then
As A to B, so C to D.

'Conversion. As AB to A, so CD to C.

Orderly Equality. If as A to B, so C to D ; and
as B to E, so D to F : then
as A to E, so C to F.

Disorderly Equality. If as A to B, so D to F, and
as B to E, so C to D : then
as A to E, so C to F.

'*Euclid's* fifth book contains but 25 propositions, to which nine more have since been added, and are commonly receiv'd. And the first six in *Euclid*, serving only for the proof of those that follow by the method of *Equimultiples*, since I intend not to make use of that method, I shall wholly omit; beginning with the Seventh, without changing either the order or number of the Propositions.

Demand

Demands, or Suppositions.

Three quantities A, B, C, being propos'd, it is requir'd to be granted, that there is a fourth possible, to which the quantity C has the same proportion, as A to B.

PROPOSITION VII.

A THEOREM.

Equal quantities have the same proportion to another quantity, and another quantity has the same proportion to quantities that are equal.

A, 8.
C, 4.
B, 8.

IF the quantities A and B be equal, they will have the same proportion to the third C.

Demonstration.

If one of the two, suppose A, had a greater proportion to C than B: A would contain any aliquot part of C, oftner than B could contain the same; and consequently A would be greater than B, which is contrary to what was suppos'd.

Again, I say, if the quantities A and B be equal, the quantity C will have the same proportion to A as to B.

IF

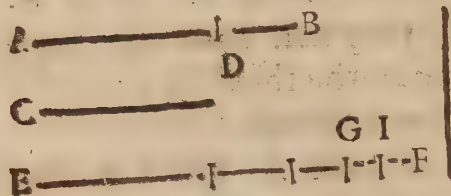
Demonstration.

If the quantity C had a greater proportion to A, than to B, it would contain a certain aliquot part of A, oftner than the like part of B, which part therefore of A must be less than the like aliquot part of B, and consequently the quantity A would be less than B, which is contrary to the supposition.

PROPOSITION VIII.

A THEOREM.

The greater of two quantities has a greater proportion to the same, than the less; and the same quantity has a lesser proportion to the greater, than to the less.



Suppose the quantities A and C be compared with the same EF, and the

AB exceeds C. I say, that AB will have a greater proportion to EF, than C will have to the same. Cut AB equal to C, and divide EF the middle, and again one half in the middle, and so on, till you come to an aliquot part EF less than DB, as GF.

Demon

4

Secondly, I say, that EF has a less proportion AB, than to C. Take a certain aliquot part, the fourth, of C, as oft as you can in EF, which suppose to be five times; either there will remain something of the quantity EF, or nothing; if nothing remain, it is evident, that five times the fourth part of AB making a greater line than so many times the fourth part of C, the fourth part of AB could not be five times contain'd in EF. But if the fourth part of C taken five times reach no farther than G, the fourth part of AB taken so many times will proceed either as far as F, or to I something short of F. If it reach as far as F, EF will have the same proportion to AB as EG to C. But (*by the preceding part*) EF has a greater proportion to AB, than EG to the same C; therefore EF has a greater proportion to C, than to AB. But if the fourth part of AB reach no farther than I, EI will have the same proportion to AB as EG to

F f

C

C. But EI has a greater proportion to C, than EG to C; therefore EF, greater than EI, has a greater proportion to C, than the same EF to AB.

PROPOSITION IX.

A THEOREM.

Quantities that have the same proportion to another quantity, or to whom another quantity has the same proportion, are equal,

A, B, C,
12.12.6

If the quantities A and B have the same proportion to a third quantity C, I say, A and B are equal.

Demonstration,

If one of the two, v. g. A were greater than B, it would have a greater proportion to the quantity C, (*by the 8.*) which is contrary to the supposition.

Secondly, if the quantity C has the same proportion to the quantities A and B; I say, A and B are equal. For if A were greater than B, C would have a greater proportion to the quantity B, than to A, (*by the 8.*) which is also contrary to the supposition.

PRC

PROPOSITION X.

A THEOREM

The quantity that has the greater proportion to another, is the greater quantity; and that the lesser, to which that other quantity has the greater proportion.

B C

IF the quantity A has a greater proportion to the quantity C, than B to the same C, I say, A is greater than B.

For if A and B were equal, they would have both the same proportion to C; and if A were less than B, B would have a greater proportion to C, than A to the same C; both which are contrary to the supposition.

Secondly, if C has a greater proportion to B than to A, I say that A will be greater than B. For if A and B were equal, C would have the same proportion to both, (*by the 7.*) And if A were less than B, C would have a less proportion to B than to A, (*by the 8.*) both which are contrary to what was suppos'd.

PROPOSITION XI.

A THEOREM.

Proportions that are equal to another, are also equal amongst themselves.

A, B; C, D; E F.
4, 2; 8, 4; 6, 3.

IF A has the same proportion to B, as C to D; and C the same proportion to D, as E to F.

I say, A will have the same proportion to B, as E to F.

Demonstration.

Since A has the same proportion to B, as C to D; A will contain any certain aliquot part of B, as oft as C contains the like aliquot part of D, (*by defin. 5.*) And in like manner, as oft as C contains that aliquot part of D, so oft will E contain the like aliquot part of F. So that so oft as A contains any certain aliquot part of B, so many times also will E contain the like aliquot part of F. Therefore A has the same proportion to B, as E to F.

PRC

PROPOSITION XII.

A THEOREM.

If many quantities be proportional, one Antecedent will have the same proportion to his consequent, as all the Antecedents taken together to all the Consequents.

A, B,
3, 12,
C, D,
2, 8.

IF A has the same proportion to B, as C to D; I say that A and C taken together will have the same proportion to B and D, as A to B.

Demonstration.

Since A has the same proportion to B, as C to D; the quantity A will contain any certain aliquot part of B, as oft as C contains the like aliquot part of D, (by defin. 5.) suppose the fourth part. Now the fourth part of B and the fourth part of D, make the fourth part of BD; and accordingly AC will contain the fourth part of BD, as oft as A contains the fourth part of B; and the like may be said of any other aliquot parts. Therefore A has the same proportion to B, as AC to BD.

PROP.

PROPOSITION XIII.

A THEOREM.

If of two equal proportions one is greater than another, the other will be so likewise.

A, B : C, D : E, F.

IF A has the same proportion to B, as C to D; but a greater proportion to B, than E to F: I say, that C also will have a greater proportion to D, than E to F.

Demonstration.

Since A has a greater proportion to B than E to F, A will contain a certain aliquot part of B, oftner than E contains the like aliquot part of F, (*by defin. 6.*) but C contains a like aliquot part of D, as oft as A contains that of B, because A has the same proportion to B, as C to D: and therefore C contains a certain aliquot part of D, oftner than E contains the like aliquot part of F; and consequently, C has a greater proportion to D, than E to F, (*by def. 6.*)

PROP

PROPOSITION XIV.

A THEOREM.

If the first quantity has the same proportion to the second, as the third to the fourth; according as the first is greater, or equal, or less, than the third, the second will be greater, or equal, or less, than the fourth.

A, B; C, D. IF A has the same proportion to B, as C to D; I say first, if A be greater than C, B will be also greater than D.

Demonstration.

Since A is greater than C, A will have a greater proportion to B, than C to the same B, (*by the 8.*) But there is the same proportion of A to B, as of C to D: therefore C has a greater proportion to D, than C to B, and consequently (*by the 10.*) B is greater than D.

Secondly, if A be equal to C, B will be also equal to D.

Demonstration.

Since A and C are equal, there will be the same proportion of A to B, as of C to the same B, (*by the 7.*) But as A to B, so C to D; therefore C has the same proportion to B, as the same C to D, and consequently B and D are equal, (*by the 9.*)

Third.

Thirdly, if A be less than C, B will also be less than D. *Demonstration.*

Since A is less than C, A will have a less proportion to B, than C to the same B, (*by the 8.*) But as A is to B, so C to D: therefore C will have a less proportion to D, than the same C to B, and consequently B will be less than D, (*by the 10.*)

PROPOSITION XV.

A THEOREM.

Equimultiples, and similar aliquot parts, are in the same proportion.

A, B;	C, D,
2, 3;	6, 9.
E, 2;	H, 3,
F, 2;	I, 3,
G, 2;	K, 3

IF the quantities C and D be the Equimultiples of A and B, their aliquot parts, A will have the same proportion to B, as C to D. Divide the quantity C into parts equal to A, *v. g.* E, F, G, and the quantity D into parts equal to B. Because C and D are the equimultiples of A and B, there will be as many parts of one, as of the other. Let the parts of D therefore be H, I, K.

Demonstration.

E has the same proportion to H, and F to I, and G to K, as A to B, because they are all equal

equal. Therefore (by the 12.) as A to B, so E, F, G, to H, I, K, i. e. so C to D.

Coroll. The same numbers of the aliquot parts of two quantities are in the same proportion that the quantities are. For since E has the same proportion to H, as C to D; and F to I, as C to D; E and F will have the same proportion to H and I, as C to D.

PROPOSITION XVI.

A THEOREM.

Alternate Proportion.

If four magnitudes of the same kind be proportional, they will be also alternatively so.

A, B; C, D,
12, 8; 9, 6,
IF A has the same proportion to B as C to D, and all the four quantities are of the same kind, that is, either all Lines, or all Superficies's, or all Solids; A will have the same proportion to C, as B to D. For if not, suppose A to have a greater proportion to C, than B to D.

Demonstration.

Since tis suppos'd, that A has a greater proportion to C than B to D, the quantity A will contain a certain aliquot part of C, v. g. a third

G g

part

part, oftner than B contains a third part of D. Let A therefore contain a third part of C four times; but B the third part of D only three times; having then divided A into four parts, each will contain one third part of C; but B being divided into four parts, they will not contain each of them a third part of D; therefore three fourths of A will contain three thirds of C, that is, the whole quantity of C; but three fourths of B will not contain three thirds, or the whole quantity of D. But on the contrary, since there is the same proportion of A to B, as of C to D, there will be also the same proportion of three fourths of A to three fourths of B, as of C to D, (*by the coroll. of the 15.*) and (*by the 14.*) if three fourths of A be equal to C, three fourths of B will be equal to D; therefore A cannot have a greater proportion to C than B to D.

LEMMA.

If the first have the same proportion to the second, as the third to the fourth; any aliquot part of the first will have the same proportion to the second, as the like aliquot part of the third to the fourth.

16,	3 ;	32,	6,
A,	B ;	C,	D,
E,		F.	
4,		8.	

If A has the same proportion to B, as C to D; and E be an aliquot part of A, and F a like aliquot part of C,

C: I say, that E will have the same proportion to B, as F to D.

Demonstration.

If E had a greater proportion to B than F to D, E would contain a certain aliquot part of B, oftner than F contains the like aliquot part of D; and consequently, E taken twice, thrice, or four times, would contain an aliquot part of B oftner than F taken twice, thrice, or four times contain'd the like aliquot part of D. But E taken four times is equal to A; and likewise F taken four times is equal to C; therefore A would contain an aliquot part of B oftner than C contain'd the like aliquot part of D, and by consequence A would have a greater proportion to B than C to D; which is contrary to the supposition.

A COROLLARY,

which Euclid places after his 4th Proposition.

Inverted Proportion.

If the first has the same proportion to the second, as the third to the fourth; the second will have the same proportion to the first, as the fourth to the third.

A, B ; C, D,
4, 8; 12, 24,
E, F,
, 3.

IF A has the same proportion to B as C to D, B will have the same proportion to A as D to C.

Demonstration.

If B had a greater proportion to A than D to C, B would contain an aliquot part of A, suppose a fourth E, oftner than D contain'd F the fourth part of C. Let us suppose then that B contains eight times the quantity E; D must contain but seven times the quantity F. Now since A has the same proportion to B as C to D, E will have the same proportion to B as F to D, (*by the preceding Lemma,*) and (*by the 15.*) E taken eight times will have the same proportion to B, as F taken eight times to D; but E taken eight times is contain'd in B, therefore F taken eight times must be contain'd in D notwithstanding what was shown to the contrary; therefore B cannot have a greater proportion to A, than D to C.

The Use.

'The Followers of *Averroes* seem to have made use of a manner of argumentation not much unlike this, to prove, that *the world had existed from eternity*; urging, that there is the same proportion between an eternal act of the will of God, and the eternal production of the world, as between a temporal act, and a temporal effect; therefore by a kind of Alternation, there is the same proportion of a temporal act of the will, *i. e.* an act begining in time to an eternal effect; as of an eternal will to a temporal effect. Now tis evident, that the

will

will, or an act of the will that begins in time, cannot produce an eternal effect; therefore the eternal act of God could not produce an effect in time. But this argument is faulty in two respects; first, in that it supposes it possible for an act of the Divine will to begin in time; and secondly, in that it is drawn from Alternate proportion, though the terms be of a different kind or species.

PROPOSITION XVII.

A THEOREM.

The Division of Proportion.

If compounded quantities be proportional, they will be so likewise being divided.

<p>AB, B; CD, D, 3; 16, 6, B; C, D, 2: 10, 6.</p>	<p>IF AB has the same proportion to B as CD to D, A will have the same proportion to B, as C to D.</p>
---	---

Demonstration.

Since AB is suppos'd to have the same proportion to B as CD to D, AB will contain a certain aliquot part of B, as oft as CD contains the like aliquot part of D. Now that aliquot part must be found as oft in B, as the like aliquot part is found

found in D. Therefore taking away B from AEB and D from CD, A will contain as many aliquot parts of B, as C contains the like of D, and consequently A will have the same proportion to B, as C to D.

PROPOSITION XVIII.

A THEOREM.

The Composition of Proportion.

If quantities, being divided, be proportional, they will be so likewise when compounded.

A, B;	C, D,
5, 3;	10, 6
AB, B;	CD, D,
8, 3;	16, 6,

IF A has the same proportion to B as C to D, A also will have the same proportion to B as CD to D.

Demonstration.

Since A is suppos'd to have the same proportion to B as C to D, A will contain any aliquot part of B, as oft as C contains the like aliquot part of D. Now the quantity B contains any of its own aliquot parts, as oft as D contains the like of his; therefore adding B to A, and D to C, AB will contain any aliquot part of B as oft as CD contains the like aliquot part of D, and consequently (*by defin. 5.*) AB will have the same proportion to B as CD to D.

A COROLLARY.

The Conversion of Proportion.

If AB has the same proportion to B as CD to D, then AB will have the same proportion to A as CD to C. For (by the preceding) A has the same proportion to B, as C to D: and (by the Coroll. of the 16.) B will have the same proportion to A, as D to C; and therefore compounding them, AB will have the same proportion to A, as CD to C.

The U.S.E.

"We have frequent use of this manner of argumentation in almost all parts of the Mathematics.

PROPOSITION XIX.

A THEOREM.

If the Wholes be in the same proportion, as the parts that are taken away from them, the Remainders will be also in the same proportion.

AB, CD ; B, D ;
6, 8 ; 4, 2 ;
A, C ; AB, CD,
2, 6 ; 16, 8,

IF the quantity AB has the same proportion to CD, as the part B to the part D: I say, A will have

have the same proportion to C, as AB to CD.

Demonstration.

AB is suppos'd to have the same proportion to CD, as B to D : therefore alternatively (according to the 16.) AB has the same proportion to B, as CD to D ; and by conversion of proportion, AB will have the same proportion to A, as CD to C ; and again alternatively, there will be the same proportion of AB to CD, of A to C.

The USE.

'This Proposition is commonly made use:
'in the rule of Fellowship. For instead of word
'ing by the rule of Three for every particular
'Associate or Partner, having done it for the
'rest, to the last they assign the Remainder
'the Gain ; supposing that if there be the same
'proportion of the whole summe of all the Princip
'cipals to the whole Gain, as of the Princip
'of one Associate to his part of the Gain : there
'will be also the same proportion of the Princip
'pal that remains to the Remainder of the
'Gain.

'The 20 and 21 Propositions are not necessary.

PRO

PROPOSITION XXII.

A THEOREM.

The proportion of Equality orderly plac'd.

If divers terms be propos'd, and an equal number of others compar'd with them, so that those which answer to each other in the same order be proportional; the firsts and the lasts will be also proportional.

2, 6, 2 ; 6, 3, 1,
A, B, C ; D, E, F.

IF the quantities A, B, C, and the quantities D, E, F be proportional; that is, if there be the same proportion of A to B as of D to E, and of B to C as of E to F; A will also have the same proportion to C, as D to F.

Demonstration.

If A has a greater proportion to C than D to F, A will contain an aliquot part, v.g. the half of C, oftner than D can contain the half of F. Let us suppose then the half of C to be contain'd twelve times in A, and the half of F only eleven times in D. Now because B has the same proportion to C as E to F, the quantity B will contain the half of C, as oft as E contains the half of F: Suppose then those halves to be con-

H h tain'd

tain'd six times in each, B and E. A, which contains the half of C twelve times, will have a greater proportion to B, which contains the same half of C six times; than D, which contains the half of F eleven times only, to E, which contains the same half of F six times; and consequently A will have a greater proportion to B, than D to E, which is contrary to what was suppos'd.

PROPOSITION XXIII.

A THEOREM

The proportion of Equality disorderly plac'd.

If two Orders of terms, be in the same proportion disorderly plac'd: the first and last of both Orders will be proportional.

A, B, C, D, E, F, G, 12, 6, 3, 8, 4, 2, 1,

IF the quantities A, B, C, D, E, F, and the others D, E, F, equal in number, be in the same proportion, disorderly plac'd, it is, if A has the same proportion to B as E to D, and B to C as D to E: A will have the same proportion to C, as D to F. Suppose B to have the same proportion to C, as F to G.

Dem.

Demonstration,

Since there is the same proportion of A to B as E to F, and of B to C as of F to G; A has the same proportion to C, as E to G, (by the 12.) Further, since B has the same proportion to C as D to E, and also as F to G; D will have the same proportion to E, as F to G, (by the 11.) and alternatively, (according to the 16.) D will have the same proportion to E, as E to G. Now we have before prov'd, that as E to G, so A to C; therefore, as A to C, so D to F.

PROPOSITION XXIV.

A THEOREM.

If the first quantity has the same proportion to the second, as the third to the fourth; and also the fifth to the second, as the sixth to the fourth: the first with the fifth will have the same proportion to the second, as the third with the sixth to the fourth.

E,	F.	IF A has the same proportion to B as C to D, and E to B as F to D; AE will have the same proportion to B, as CF to D.
4,	6.	
5, 2; 9, 3,		
A, B; C, D,		

Demonstration.

Since A has the same proportion to B as C to D,
Hh 2 D,

D, A will contain any aliquot part of B, as oft as C contains the like aliquot part of D, (*by def. 5.*) In like manner, E will contain the same aliquot part of B, as oft as F contains the like aliquot part of D; so that A and E will contain any aliquot part of B, as oft as C and F contain the like aliquot part of D: therefore AE will have the same proportion to B, as CF to D.

PROPOSITION XXV.

A THEOREM.

If four magnitudes be proportional, the greatest and the least will exceed the other two.

B, D,
8, 6,
4, 3; 4, 3,
A, C; E, F,

IF the four magnitudes A, B, C, D, E, F, be proportional, and AB the greatest and F the least; AB and F will exceed CD and E. Since AB has the same proportion to CD as E to F, and AB suppos'd greater than E; CD will be also greater than F, (*by the 14.*) Divide therefore A and CD so, that the magnitude A may be equal to H, and CD so, that the magnitude C may be equal to F.

Demonstration.

Since AB has the same proportion to CD as A to C, B will also have the same proportion to D as AB to CD, (*by the 19.*) and AB being

ful

suppos'd greater than CD, B will be greater than D. Now if to A and E, which are equal, be added F and C, which are also equal, A and F will be equal to C and E, and adding to the two first B, which is the greater, and to the two last D, which is the less, AB and F will be greater than CD and E.

The U S E.

By this Proposition is demonstrated a propriety of *Geometrical* proportionality, whereby it is distinguish'd from that which is call'd *Arithmetical*. For in this latter the two middle terms are equal to the two extremes; but in the former, (as has been prov'd,) the greatest and the least exceed the two others.

'Tho the nine following Propositions are not *Euclid's*; yet, because many make use of them, and quote them as if they were his, I thought I ought not to omit them.

PROP.

PROPOSITION XXVI.

A THEOREM.

If the first has a greater proportion to the second than the third to the fourth, the fourth will have a greater proportion to the third than the second to the first.

9, 4 ; 6, 3,
A, B ; C, D,
E.
3.

IF A has a greater proportion to B than C to D, D will have a greater proportion to C than B to A. Suppose E to have the same proportion to B as C to D.

A will be greater than E, (by the 10.)

Demonstration.

There is the same proportion of E to B, as C to D : therefore (by the Coroll. of the 16.) E has the same proportion to C, as B to D. But B has a greater proportion to E than to A, (by the 8.) therefore D has a greater proportion to C than B to A.

PROP

PROPOSITION XXVII.

A THEOREM.

If the first has a greater proportion to the second than the third to the fourth, the first will also have a greater proportion to the third than the second to the fourth.

9, 4; 6, 3,
A, B; C, D,
E.
8.

IF A has a greater proportion to B than C to D, I say, that A will have a greater proportion to C than B to D. Let E have the same proportion to B, as C to D; in that case A must be greater than E.

Demonstration.

E has the same proportion to B, as C to D: therefore (by the 16.) E has the same proportion to C, as B to D. And because A is greater than E, the proportion of A to C will be greater than that of E to C. Therefore the proportion of A to C, is greater than that of B to D.

PROP.

PROPOSITION XXVIII.

A THEOREM.

If the first has a greater proportion to the second than the third to the fourth, the first and the second will have a greater proportion to the sixth than the third and the fourth to the fourth.

9, 4 ; 6, 3 ,
A, B ; C, D,
E,
8.

IF the proportion of A to B be greater than that of C to D, the proportion of AB to B will also be greater than that of CD to D. Suppose E to have the same proportion to B, as C to D.

Demonstration.

E has the same proportion to B, as C to D, therefore (by the 18.) EB has the same proportion to B, as CD to D. And AB being greater than EB, AB will have a greater proportion to B, than EB to the same B, and consequently than CD to D.

PRO

PROPOSITION XXIX.

A THEOREM.

If the first with the second has a greater proportion to the second, than the third with the fourth to the fourth; the first will have a greater proportion to the second, than the third to the fourth.

9, 4; 6, 3.
A, B; C, D,
E.
8.

IF the proportion of AB to B be greater than the proportion of CD to D, the proportion of A to B will be also greater than that of C to D. Suppose EB to have the same proportion to B, as CD to D: EB will be less than AB, and E less than A.

Demonstration.

Since EB has the same proportion to B, as CD to D; dividing them E will have the same proportion, to B as C to D, (*by the 17.*) And A being greater than E, the proportion of A to B will be greater than that of E to the same B, and consequently than that of C to D.

PROPOSITION XXX.

A THEOREM.

If the proportion of the first with the second to the third, be greater than that of the third with the fourth to the fourth; the proportion of the first with the second to the first, will be less than that of the third with the fourth to the third.

A, B; C, D,
2, 4; 6, 3.

IF the proportion of AB to B be greater than that of CD to D, the proportion of AB to A will be less than that of CD to C.

Demonstration.

The proportion of AB to B is suppos'd to be greater than that of CD to D: therefore (*by the 29.*) the proportion of A to B will be greater than that of C to D; and (*by the 26.*) the proportion of D to C will be greater than that of B to A: therefore being compounded (*by the 28.*) the proportion of CD to C, will be greater than that of AB to A.

PROPOSITION XXXI.

PROPOSITION XXXI.

A THEOREM.

If many quantities are in a greater proportion among themselves, than an equal number of other quantities, plac'd after the same manner; the first of the first order will be in a greater proportion to the last of that order, than the first of the second order to the last of that.

A, B, C, | D, E, F,
6, 10, 3, | 9, 6, 2,

IF A has a greater proportion to B, than D to E; and B a greater proportion to C, than E to F; A will have a greater proportion to C, than D to F.

Demonstration.

Since A has a greater proportion to B than D to E, A will also have a greater proportion to C than B to E; and because B has a greater proportion to C than E to F, B will also have a greater proportion to E than C to F. Therefore A will have a greater proportion to D than C to F: and Alternatively (*by the 27.*) A will have a greater proportion to C, than D to F.

PROPOSITION XXXII.

A THEOREM.

If many quantities are in greater proportion among themselves, than an equal number of other quantities, plac'd after a different manner; the first of the first order will have a greater proportion to the last of that order, than the first of the second order to the last of that.

13, 6, 2,		
A, C, E,		
B,	F	H, I, K,
12,	3	4, 2, 1,

IF A has a greater proportion to C than to K, and C a greater proportion to E than to I; A will have a greater

proportion to E, than H to K. Suppose to have the same proportion to C as I to K, and C the same proportion to F as H to I; then will be greater than B, and F than E.

Demonstration.

Since tis suppos'd that B has the same proportion to C as I to K, and C to F as H to I, B will have the same proportion to F, as H to K, (*by the 23.*) But A has a greater proportion to F, than B to the same F, (*by the 8.*) and the proportion of A to E is greater than that of A to F, because F is greater than E: therefore the proportion of A to E, is greater than that of H to K.

PRO

PROPOSITION XXXIII.

A THEOREM.

If the Whole has a greater proportion to the Whole than the part taken away to the part taken away, the Remainder will have a greater proportion to the Remainder than the Whole to the Whole.

13, 4; 6, 2,
A, B; C, D.

IF AB has a greater proportion to CD than B to D, A will have a greater proportion to C than AB to CD.

Demonstration.

We suppose that the proportion of AB to CD is greater than that of B to D; therefore (*by the 27.*) the proportion of AB to B is greater than that of CD to D: and (*by the 30.*) the proportion of AB to A, is less than that of CD to C; therefore alternatively, the proportion of AB to CD is less than that of A to C.

PROP.

PROPOSITION XXXIV.

A THEOREM.

If two orders of magnitudes be propos'd, and the proportion of the first of the first, to the first of the second, be greater than that of the second to the second; and that greater than that of the third to the third, &c. the whole first order will have a greater proportion to the whole second, than the whole first order except its first magnitude to the whole second order except its first magnitude. But a less proportion than that of the first magnitude of the first order, to the first magnitude of the second; and lastly, a greater proportion, than that of the last magnitude of the first order, to the last of the second.

12, 6, 4,	4, 2, 3,
A, B, C,	E, F, G,

IF the proportion of A to E be greater than that of B to F, and the proportion of B to F greater than that of C to G: I say, that A, B, C, will have a greater proportion to E, F, G, than the proportion of BC to FG.

Demonstration.

Tis suppos'd the proportion of A to E is greater than that of B to F; and therefore alternately

Alternatively, the proportion of A to B is greater than that of E to F; and by compounding them, the proportion of AB to B greater than that of EF to F; and again alternatively, the proportion of AB to EF greater than that of B to F. And because the proportion of the whole AB to EF is greater than that of the part B to the part F, the proportion of the Remainder A to the Remainder E will be greater than that of the whole AB to the whole EF. In like manner, I may prove the proportion of B to F greater than that of BC to FG, and consequently that of A to E much greater than that of BC to FG. Therefore alternatively, the proportion of A to BC is greater than that of E to FG; and compounding them, the proportion of A, B, C, to BC, greater than that of E, F, G, to FG: therefore the proportion of A, B, C, to E, F, G, will be greater than that of BC to FG. Secondly, the proportion of A to E, is greater than that of A, B, C, to E, F, G.

Demonstration.

I have demonstrated, that the proportion of the whole A, B, C, to the whole E, F, G, is greater than that of the part BC to the part FG: therefore the proportion of the Remainder A to the Remainder E, will be greater than that of the whole A, B, C, to the whole E, F, G, (by *e 33.*)

Thirdly, the proportion of A, B, C, to E, F, G, is greater than that of C to G, *De-*

Demonstration.

The proportion of A to E is greater than that of B to F ; and therefore alternatively that of A to B is greater than that of E to F ; and compounding them, the proportion of AB to B will be greater than that of EF to F ; and again alternatively, that of AB to EF greater than that of B to F . But the proportion of B to F is greater than that of C to G , therefore the proportion of AB to EF is greater than that of C to G ; and that of AB to C greater than that of EF to G ; and therefore by compounding them, the proportion of A, B, C , to C , will be greater than that of E, F, G , to G ; and that of A, B, C , to E, F, G , greater than that of C to G .

THE SIXTH BOOK

OF THE

ELEMENTS

OF

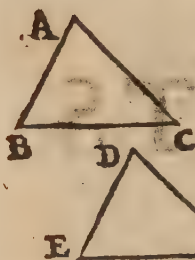
EUCLID.

THis Book begins to apply the doctrine of Proportions, explain'd in general in the preceding, to particular matters; and taking its rise from the most simple figures, *i. e.* Triangles, it gives Rules to determine not only the proportion of their sides, but also that of their capacity, *Area* or superficies. In the next place we learn from it, how to find out proportional lines, and to augment or diminish any figure, according to any proportion assign'd. Here also is demonstrated the most useful Rule of Three; and the Forty-seventh of the First extended to any figure whatsoever. Lastly, it lays down the most facile and most certain principles to conduct us in taking any manner of Dimensions.

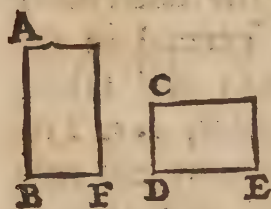
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DE-

DEFINITIONS.



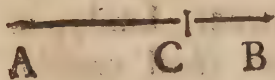
1. **R**ECILINEAL figures are similar, when their angles are equal, and the sides that form those angles, proportional. "As the triangles ABC , DEF , will be similar if the angles A and D , B and E , C and F , be equal; and AB has the same proportion to AC as DE to DF , and AB to BC as DE to EF .



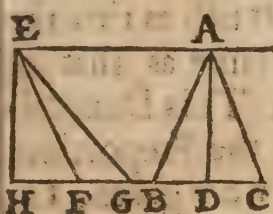
2. Figures are reciprocal when they may be so compared, that the antecedent of one proportion and the consequent of another are both found in the same figure.

That is, when the Analogy begins and ends in the same figure. As if AB has the same proportion to CD , as DE to BF .

3. A line is divided according to the extreme and middle proportion, when the whole line has the same proportion to the greater part, as the greater part to the less. "As AB has the same proportion



to AC, as AC to CB : the line AB is divided according to the extreme and middle proportion.



4. The height of any figure is a perpendicular drawn from its summit to its base.

As in the triangles ABC, EFG, the perpendiculars

EH and AD, whether they

fall within or without the triangles, are their heights. Hence it follows, that all triangles

and parallelograms, that have equal heights,

may be plac'd within the same parallels. For

having set their bases upon the line HC, if the

perpendiculars DA and HE be equal, the lines

EA and HC will be parallels.

5. A Proportion is said to be compounded of many others, when the

*quantities of those proportions being multiply'd, make

another. To understand the true intent of

this definition, it must be observ'd, that every

Proportion, at least, every *ra-

tional Proportion, takes its De-

nomination from a certain num-

ber, denoting that respect or relation that the

antecedent of the proportion bears to the con-

sequent. As if two magnitudes were propos'd,

one of twelve foot in length, and the other of

six ; we should call that proportion of 12 to

* Denominators.
* Expressible, by
true numbers.

'6 the double proportion. In like manner if
 '4 and 12 were propos'd, we should give that
 'the name of subtriple proportion, $\frac{1}{3}$ being its
 'Denominator ; importing as much as that the
 'proportion of 4 to 12, is the same as that of
 ' $\frac{1}{3}$ to unity, or as one to three. This Denomi-
 'nator is call'd the quantity of the Proportion.
 'Suppose therefore three terms were given, 12,
 '6, and 2 ; the first proportion of 12 to 6 being
 'double, its Denominator is two ; the second of
 '6 to 2 being triple, its Denominator is three ;
 'the proportion therefore of 12 to 2 is said to
 'be compounded of that of 12 to 6, and of 6
 'to 2, the double and the triple proportion.
 'To find therefore the Denominator of the pro-
 'portion of 12 to 2, multiply three by two, and
 'the product six will show the proportion of
 '12 to 2 to be sextuple, *i. e.* as one to six. This
 'is that which *Mathematicians* commonly un-
 'derstand by compounded Proportion, though
 'methinks it might more properly have been
 'call'd Proportion multiply'd.

PROP.

PROPOSITION I.

A THEOREM.

Parallelograms, and triangles, of the same height, are in the same proportion as their bases.



Suppose the triangles AGC, and DEM, to have the same height, and to be plac'd between the same parallels, AD, and GM: I say, they

ll have the same proportion as their bases C, and EM. Divide the base EM into as many equal parts as you please, and draw lines from the point D to each division, as DF, DH, L. In like manner divide the line CG into parts equal to those of the line EM, and draw lines from its summit A to those divisions, as AB, AI, &c. All those little triangles, being clos'd within the same parallels, and having equal bases, are equal, (*by the 38. 1.*)

Demonstration.

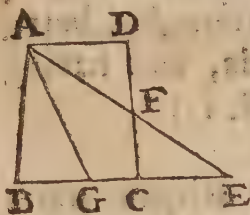
The base GC contains so many aliquot parts EM, as there are parts found in it equal EF; but as many parts equal to EF as are found in the base GC, so many little triangles contain'd in the triangle AGC, equal to those

those contain'd in DEM; which being equal among themselves are the aliquot parts of the triangle DEM. As oft therefore as the base GC contains those aliquot parts of EM, so oft does the triangle AGC contain the aliquot parts of the triangle DEM; which also will happen in every division whatsoever: therefore the triangle AGC has the same proportion to the triangle DEM, as the base GC to the base EM.

Now Parallelograms, describ'd upon the same bases, and enclos'd between the same parallels, are double the triangles, (*by the 41. Prop.*) therefore they are in the same proportion as the triangles, *i. e.* as their bases.

The USE.

This Proposition is not only serviceable in demonstrating those that follow, but also of great use in dividing large fields, or plaines. As for example, suppose you were to take the third part of the Trapezium ABCD, having the sides AD and BC parallel; produce the line BC to E, so that CE may be equal to AD; and taking BG the third part of BE, draw AG: I say the angle ABG is the third part of the Trapezium ABCD.



Dem.

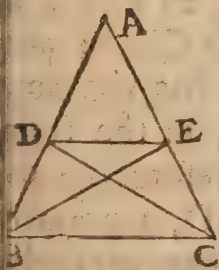
Demonstration.

"The triangles ADF, and FCE, are equiangular, because the lines AD and CE are parallels, and their sides AD and CE are equal: therefore (by the 26. 1.) the triangles are equal, and consequently the triangle ABE is equal to the *trapezium* ABCD. But the triangle ABG is the third part of the triangle ABE, (by the preceding,) therefore the triangle ABG is the third part of the *trapezium* ABCD,

PROPOSITION II.

A THEOREM.

A line drawn in a triangle parallel to its base divides its sides proportionally; and the line that divides the sides of a triangle proportionally, will be parallel to its base.



IF in the triangle ABC the line DE be parallel to the base BC, the sides AB and AC will be divided proportionally, i. e. AD will have the same proportion to DB, as AE to EC. Draw the lines DC and BE.

The triangles DBE, and DEC, having the same base

base DE, and being enclos'd within the same parallels DE and BC, are equal, (*by the 37. 1.*)

Demonstration.

The triangles ADE and DBE have the same height E, taking AD and DB for their bases, and therefore may be plac'd within the line AB and another parallel to it drawn through the point E, and consequently have the same proportion as their bases, (*by the 1.*) i.e. the triangle ADE has the same proportion to the triangle DBE or its equal CED, as AD to DB. But the triangle ADE has likewise the same proportion to the triangle CED, as AE to AC; and therefore AD has the same proportion to DB, as AE to EC.

Secondly, suppose AE to have the same proportion to EC, as AD to AB: I say the lines DE and BC will be parallels.

Demonstration.

AD has the same proportion to DB, as the triangle ADE to the triangle DEB, (*by the 1.*) and AE has the same proportion to EC, as the triangle ADE to the triangle CED; and consequently the triangle ADE has the same proportion to the triangle DEB, as the triangle ADE to the triangle CED. Therefore (*by the 9. 5.*) the triangles BDE, and CED, are equal, and (*by the 39. 1.*) between the same parallels. Therefore the lines DE and BC are parallels.

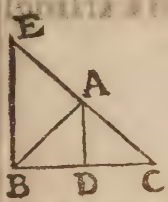
The USE.

This Proposition is absolutely necessary for the demonstration of those that follow. It may also be serviceable in taking Dimensions. As, in the following figure, if you were to measure the height of BE; having erected a staff or pole DA, there will be the same proportion of CD to DA, as of BC to BE.

PROPOSITION III.

A THEOREM.

A line that divides an angle of a triangle into two equal parts, divides its base into two parts which have the same proportion as the sides. And if it divide the base into two parts proportional to the sides, it will divide the angle into two equal parts.



IF the line AD divide the angle BAC into two equal parts, AB will have the same proportion to AC as BD to DC. Produce the side CA, and take AE equal to AB; then draw the line EB.

Demonstration.

The external angle CAB is equal to the two internal angles AEB, and ABE: which being equal, (by the 5. 1.) because the sides AE and

LI

AB

AB are equal ; the angle BAD, which is the half of BAC, will be equal to one of them, suppose ABE ; therefore (*by the 27. 1.*) the lines AD and EB are parallel, and (*by the 2.*) there is the same proportion of EA or AB to AC, as CD to BD to DC.

Secondly, if AB has the same proportion to AC as BD to DC, the angle BAC will be divided into two equal parts.

Demonstration.

AB or EA has the same proportion to AC as BD to DC : therefore EB and AD are parallel ; and (*by the 29. 1.*) the alternate angles EBA and BAD, the internal BEA, and the external DAC, will be equal : and the angles BEA and EBA being equal, the angles BAD and DAC will be also equal. Therefore the angle BAC will be divided into two equal parts.

The Use.

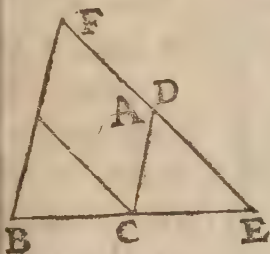
‘ We make use of this Proposition chiefly to find the proportion of the sides of a triangle.

PROP.

PROPOSITION IV.

A THEOREM.

The sides of equiangular triangles are proportional.



IF the triangles ABC, and DCE, be equiangular, *i.e.* if the angles ABC and DCE, BAC and CDE be equal, AB will have the same proportion to BC as DC to CE, and also AC will have the same proportion to CB as DE to EC. Joyn the triangles after such a manner, that the bases BC and CE may be upon the same line, and produce the sides BA and ED till they meet in F; since the angles ACB and DEC are equal, the lines AC and FE are parallels, [*by the 28. 1.*] and by the same reason CD and BF are parallels, and therefore AFDC a Parallelogram.

Demonstration.

In the triangle BFE, AC is parallel to FE, therefore [*by the 2.*] AB has the same proportion to AF, or, which is equal to it, CD, as BC to CE: and alternatively, AB has the same proportion to BC, as CD to CE. In like manner in the same triangle, CD being parallel to

L 1 2

BF,

BF, FD or AC has the same proportion to DM as BC to CE, [by the 2.] and alternatively, AD has the same proportion to BC, as DE to CE..

The U S E.

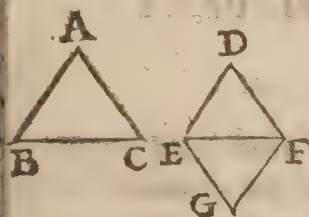
'This Proposition is of so general use, that it may pass for a most universal principle in treating all manner of Dimensions. For in the first place all the methods of measuring inaccessible lines, by describing a small triangle similar to that which is form'd upon the ground, founded upon it; as also the greatest part of those *Mathematical Instruments*, upon which are describ'd triangles, similar to those of which we desire to take the dimensions, as the *Geometrical Square*, the *Pantometer*, the *Arbaletr* or *Cross-staff*, &c. Nor could we know how to raise the Plane of any place, but by the help thereof. So that in fine, to unfold all the use of this Proposition, it would be necessary to transcribe the whole first book of *Practical Geometry*.

PROI

PROPOSITION V.

A THEOREM.

Triangles, which have proportional sides, are equiangular.



IF the triangles ABC and DEF have proportional sides, *i.e.* if AB has the same proportion to BC as DE to EF, and also AB the same proportion to AC as DE to

DF; the angles ABC and DEF, A and D, C and F, will be equal. Make the angle FEG equal to the angle B, and EFG equal to the angle C.

Demonstration.

The triangles ABC and EFG have two angles equal, therefore (*by Coroll. 2. of the 32. 1.*) they are equiangular; and (*by the 4.*) AB has the same proportion to BC as GE to EF. Now 'tis suppos'd, that DE has the same proportion to EF as AB to BC, therefore DE has the same proportion to EF as EG to EF; and consequently (*by the 9. 5.*) DE and EG are equal. After the same manner DF may be prov'd equal to FG, and consequently (*by the 8. 1.*) the triangles DEF and GEF are equiangular. But the angle

angle GEF was made equal to B, therefore the angle DEF is equal to the angle B, and the angle DFE to the angle C; and consequently, the triangles ABC and DEF are equiangular.

PROPOSITION VI.

A THEOREM.

Triangles, which have each one of their angles equal to one of the other, and the sides containing that angle proportional, are equiangular.

IF the triangles ABC and DEF (see fig. preceding) have the angles B and E equal, and the side AB has the same proportion to BC as DE to EF, the triangles ABC and DEF will be equiangular. Make the angle FEG equal to the angle B, and the angle EFG equal to the angle

Demonstration.

The triangles ABC and GEF are equiangular, (by Coroll. 2. of the 32. 1.) therefore AB has the same proportion to BC as GE to EF (by the 4.) But as AB to BC, so is DE to EF, therefore DE has the same proportion to EF as GE to the same EF; and therefore (by the 5.) DE and EG are equal; and the triangles DEF and GFE, having the angles DEF and GEF, each equal to the angle B, and the

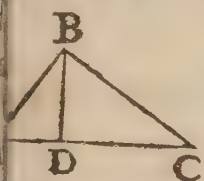
DE and EG equal, with the side EF common to both, will be equal in all respects, (*by the 4. of the 1. book.*) therefore they will be equiangular; and the triangle EGF being equiangular to ABC, the triangles ABC and DEF are equiangular.

The Seventh Proposition is of no use.

PROPOSITION VIII.

A THEOREM.

A perpendicular, drawn from the right angle of a rectangular triangle to the opposite side, divides the triangle into two others similar to it,



IF the perpendicular BD be drawn from the right angle ABC to the opposite side AC, it will divide the rectangular triangle ABC into two triangles ADB and BDC, which will be similar, or equiangular to the triangle ABC.

Demonstration.

The triangles ABC and ADB have the same angle A, and the angles ABC and ADB right angles, therefore they are equiangular, [*by Coroll. 2. of the 32. 1.*] In like manner the triangles BDC and ABC have the angle C common to both, and the angles ABC and BDC right angles; therefore they also are equiangular.

There-

Therefore the triangles ABC , BDC , and ADB are similar triangles.

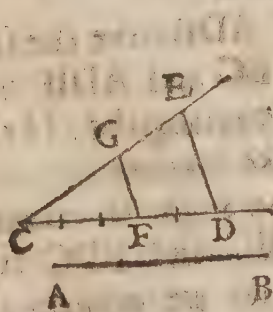
The Use.

‘By the help of this Proposition inaccessible distances may be measur’d by a Carpenter’s square. As for example, if I were to measure the distance DC ; having drawn the perpendicular BD , and plac’d my square upon the point B in such a manner, that by one of its sides BC I could observe the point C , and by the other the point A : it is evident, there would be the same proportion of AD to DB as of DB to DC . Therefore multiplying DB by its self, and dividing the product by AD the Quotient would be DC .

PROPOSITION IX.

A PROBLEM.

Of a line given to cut off what part you please.



LET the line propos’d AB , from which you are to take away three fourths. Make an angle EC and upon one of its sides take five equal parts, three of which shall be contain’d

CF: then taking CE equal to AB, draw the line DE, and another parallel to that FG: the line CG will contain three fifth parts of CE or AB.

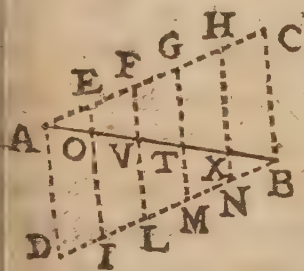
Demonstration.

In the triangle ECD, FG being parallel to the base DE, CF will have the same proportion to FD as CG to GE, [by the 2.] and compounding them, [by the 18. 5.] CE will have the same proportion to CG as CD to CF; and (by the Coroll, of the 16. 5.) CG will have the same proportion to CE as CF to CD. But CF contains three fifths of CD, therefore CG will contain three fifths of CE or AB.

PROPOSITION X.

A PROBLEM.

To divide a line after the same manner as another line given is divided.



IF you would divide the line AB after the same manner as AC is divided, make with the two lines the angle CAB of what magnitude you please; then draw the line BC, and parallel to Mm it

it the lines EO, FV, and the rest. The line AE will be divided after the same manner that AC is.

Demonstration.

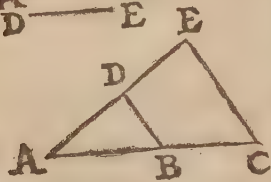
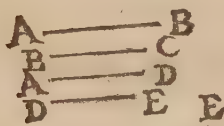
Since in the triangle BAC, HX is drawn parallel to the base BC, it will divide the sides AB and AC proportionally, (*by the 2.*) and the same may be said of all the rest.

To do this more easily, you may draw the line BD parallel to AC; and transferring the divisions of AC to BD, draw lines from one to the other.

PROPOSITION XI.

A PROBLEM.

Two lines being given to find a third proportional



IF you would find a third proportional to the lines AB and BC, *i.e.* that there may be the same proportion of AB to BC, as of BC to the line sought; make at pleasure the angle EAC, and upon one of its sides take

the lines AB and BC, one immediately after the other; and upon the other side take AD equal to BC: then draw the line BD, and parallel to it the line CE; and the line DE will be that which you seek.

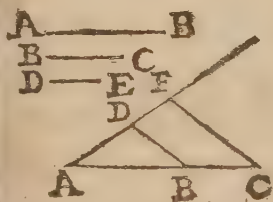
Demonstration.

In the triangle EAC the line DB is parallel to the base CE, therefore [by the 2.] there is the same proportion of AB to BC, as of AD or BC to DE.

PROPOSITION XII.

A PROBLEM.

Three lines being given to find a fourth proportional.



LET the three lines propos'd, to which you are to find a fourth proportional, be AB, BC, and DE. Make at pleasure the angle FAC, and take upon AC the lines AB

and BC, and upon AF the line AD equal to DE; then draw the line DB, and parallel to it FC: I say, that DF is the line sought, i. e. there is the same proportion of AB to BC as of DE or AD to DF.

Demonstration.

In the triangle FAC the line DB is parallel to the base FC: there is therefore the same proportion of AB to BC, as of AD to DF, [by the 2.]

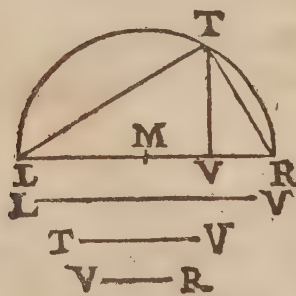
The USE.

'The use of the *Compass* of *Proportion*
 'grounded upon these four Propositions ; for
 'that Instrument teaching us to divide a line as
 'we please ; to make use of the *Rule of Three*
 'without the help of *Arithmetick* ; to extract
 'the Square, and Cubick roots ; to double the
 'Cube ; to measure all sorts of *Triangles* ; to
 'find the capacity of *Superficies*'s, and the solidity
 'of bodies ; and to augment or diminish
 'any figure, according to what proportion we
 'desire ; all these operations are demonstrated
 'by the preceding Propositions.

PROPOSITION XIII.

A PROBLEM.

Two lines being given to assign a middle proportional.



IF you desire a middle
 proportional between
 the lines LV, and VR
 having plac'd them
 that they make but one
 right line LR, dividing
 that line into two equal
 parts in M ; and ha

ing describ'd a semicircle LTR from the center M, draw the perpendicular VT, which will be a middle proportional between LV and VR. Draw the lines LT, and TR.

Demonstration.

The angle LTR, being describ'd in a semicircle, is a right angle, (*by the 31. 2.*) and (*by the 8.*) the triangles LVT and TVR are similar; therefore there is the same proportion of LV to VT in the triangle LVT, as of VT to VR in the triangle TVR, (*by the 4.*) therefore VT is a middle proportional between LV and VR.

The U S E.

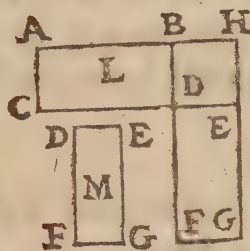
' By this Proposition any rectangular Parallelogram may be reduc'd to a Square. For example, in the Rectangle contain'd under LV and VR, the square of VT is equal to the Rectangle under LV and VR; as I shall hereafter demonstrate.

PROP.

PROPOSITION XIV.

A THEOREM.

Equal equiangular Parallelograms have their sides reciprocal; and equiangular Parallelograms, that have their sides reciprocal, are equal.



IF the parallelogramss and M be equiangular and equal, their sides will be reciprocal, *i. e.* C will have the same proportion to DE as FD to DB. Since they have

equal angles, they may be so joyn'd together that the sides CD and DE, FD and DB, will concur in two right lines, (*by the coroll. of 15. 1.*) producing therefore the sides AB and GE, you complet the parallelogram BDEH.

Demonstration.

The parallelograms L and M being equal will have the same proportion to the parallelogram BDEH. But the proportion of L to BDEH, is as the base CD to the base DE; and that of M, or DFGE, to BDEH, is as the base FD to the base DB, (*by the 1.*) Therefore C has the same proportion to DE, as FD to DB.

Secondly, if the Parallelograms L and M be equiangular, and have their sides reciprocal, they will be equal.

Demonstration.

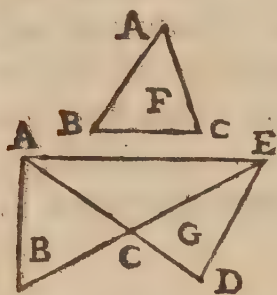
The sides of the parallelograms are suppos'd to be reciprocal, *i. e.* that there is the same proportion of CD to DE, as of FD to DB: but as the base CD is to DE, so is the parallelogram L to the parallelogram BDEH, (*by the 1.*) and as FD to DB, so is the parallelogram M to BDEH; therefore L has the same proportion to BDEH, as M to the same BDEH; therefore (*by the 9. 5.*) the parallelograms L and M are equal.

PROP.

PROPOSITION XV.

A THEOREM.

Equal triangles, that have one angle equal each to the other, have the sides, that form that angle reciprocal: and if those sides be reciprocal, those triangles will be equal;



IF the triangles F and C being equal, have the angles ACB and DCE equal, their sides that form those angles will be reciprocal, *i.e.* BC will have the same proportion to CD as CD to CA . Place the triangles so, that the sides CD and CA make one right line; and then because the angles ACB and DCE are suppos'd to be equal, BC and CE will also make one right line, (*Coroll. of the 15. 1.*) Draw the line AE .

Demonstration:

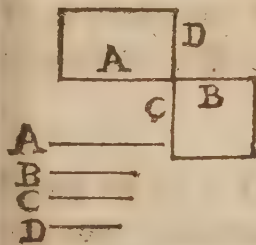
The triangle ABC has the same proportion to the triangle ACE , as the triangle ECD , equal to the former, to the same ACE , (*by the 7. 5.*) But as ABC to ACE , so is the base BC to the base CE , [*by the 1.*] having both the same height A ; and as ECD to ACE , so is the base

CD to CA, (*by the same* :) therefore BC has the same proportion to CE, as CD to CA. But if the sides be suppos'd reciprocal, *i. e.* that BC has the same proportion to CE as CD to CA, the triangles ABC and CDE will be equal, because they will both have the same proportion to ACE.

PROPOSITION XVI.

A THEOREM.

If four lines be proportional, the rectangle contain'd under the first and the fourth, will be equal to the rectangle contain'd under the second and the third. And if the rectangle contain'd under the extremes be equal to that contain'd under the middle termes, the four lines will be proportional.



IF the lines A, B, C, D, be proportional, *i. e.* if as A to B, so C to D, the rectangle contain'd under the first A, and the fourth D, will be equal to the rectangle contain'd under B and C.

Demonstration.

The rectangles have one angle equal each to th' other, because tis a right angle in both; their sides also are reciprocal: therefore they are equal, [*by the 14.*]

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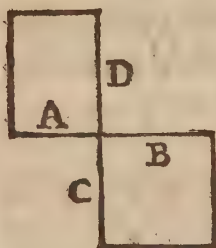
In

In like manner, if they are equal, their sides will be reciprocal, *i.e.* A will have the same proportion to B as C to D.

PROPOSITION XVII.

A THEOREM.

If three lines be proportional, the rectangle contain'd under the first and the third, will be equal to the square of the middle terme. And if the square of the middle terme be equal to the rectangle under the extremes, the three lines are proportional.



IF the three lines A, B, D, be proportional, the rectangle contain'd under A and D will be equal to the square of B. Take C equal to B, and then will be the same proportion A to B as of C to D; therefore

the four lines are proportional.

Demonstration.

The rectangle under A and D will be equal to that under B and C, (*by the preceding,*) but the last rectangle is a square, the lines B and C being equal: therefore the rectangle contain'd under A and D is equal to the square of B.

In like manner, if the rectangle under A and

D be equal to the square of B, A will have the same proportion to B as C to D: and B and C being equal, A will have the same proportion to B, as B to D.

The USE.

' By these four propositions may be demonstrated that Rule in *Arithmetick*, which is 'commonly call'd the *Rule of Three*; and consequently, the Rules of *Fellowship*, of *False*, and 'all those others that depend upon Proportion. 'For example, suppose Three numbers given, 'A 8, B 6, and C 4, and it be requir'd to find 'a fourth proportional number; which taking 'as found, I will call D. The rectangle then 'contain'd under A and D, is equal to that under B and C. But I may have this latter rectangle by multiplying B by C, *i. e.* six by four, 'the product will be twenty four; therefore the 'rectangle contain'd under A and D is also 'twenty four; and therefore dividing that number by A, which is 8, the Quotient three will 'be the number sought.

N^o 2

PROP.

PROPOSITION XVIII.

A PROBLEM.

To describe a Polygon similar to another upon a line given.



LET AB be the line assign'd, and upon which you are required to describe a Polygon similar to the Polygon CFDE; and having divided the polygon CFDE into triangles, upon the line AB make triangle ABH similar to the triangle CFE, i. e. make the angle ABH equal to the angle CFE, and BAH equal to FCE, for then the triangles ABH and CFE will be equiangular, [by Corol. of the 32.1.] Make also upon the line BH a triangle HBG equiangular to FDE.

Demonstration.

Since the triangles, which are part of the polygons, are equiangular, the two polygons are equiangular. Further, since the triangles ABH and CFE are equiangular, AB will have the same proportion to BH as CF to FE, [by the 4th Prop.] In like manner, the triangles HBG and FED being equiangular, BH will have the same proportion to BG as FE to FD: and by equality (accord-

[according to defin. 18.5.] AB will have the same proportion to BG, as CF to FD. And the same may be said of all the other sides. Therefore (by defin. 1.) the polygons are similar.

The USE.

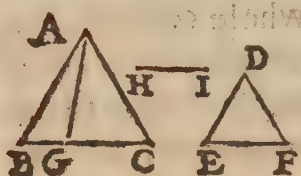
'Upon this proposition is grounded the greatest part of *Practical Geometry*, that relates to the raising the plane of any place, as of a building, field, forrest, or a whole country. For having divided a line into equal parts, to answer the feet or yards contain'd in the plane, you may describe a figure similar to, but less than, the Original, in which you may see the proportion of all its lines. And having by experience found it much more easie to travel upon paper, than to take a tedious journey either by land or water, this proposition will likewise afford us assistance in this respect, in forming us in almost all the parts of *Geodesia*, and *Chorography*; and giving instructions how to compose *Geographical Charts*, and *Maps*; which are nothing else but methods of reducing great figures to small. Further, the use of this proposition extends it self to almost all those Arts, that require the design and model of their works before hand.

PROP.

PROPOSITION XIX.

A THEOREM.

*Similar triangles are in the duplicate proportions
their homologous sides.*



IF the triangles ABC and DEF be similar or equiangular, they will be in the duplicate proportion their homologous sides BC and EF, *i. e.* the proportion of the triangle ABC to the triangle DEF will be the duplicate of the proportion of BC to EF ; so that finding a third proportional HI to the lines BC and EF, and making BC to have the same proportion to EF as EF to HI, the triangle ABC will have the same proportion to DEF as the line BC to the line HI ; which is to have to it a duplicate proportion, [*by defin. 11.5.*] Take BG equal to HI, and draw the line AG,

Demonstration.

The angles B and E of the triangles ABC and DEF are equal ; and besides, since the triangles ABC and DEF are similar, AB will have the same proportion to DE as BC to EF, [*the fourth.*] But as BC to EF, so EF to HI or BG ; therefore as AB to DE, so EF to BG.

and consequently, the sides of the triangles ABG and DEF being reciprocal, the triangles will be equal, [by the 15.] And [by the 1.] the triangle AHC has the same proportion to the triangle ABG, as BC to BG or HI: therefore the triangle ABC has the same proportion to the triangle DEF, as BC to HI.

The Use.

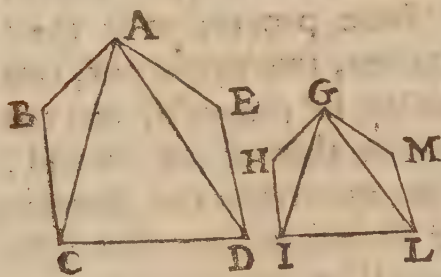
These Propositions may help to correct the error of those, who are apt to imagine similar figures to have the same proportion as their sides. For if two squares, two pentagons, two hexagons, or two circles, be propos'd, and the side of the first be double that of the second, the first figure will be quadruple the second: if the side of the first be triple that of the second, the first figure will be nine times greater than the second. Therefore to make a square triple to another, you must seek a middle proportional between one and three, and you'll find for the side of your triple figure almost $1\frac{1}{2}$.

PROP.

PROPOSITION XX.

A THEOREM.

Similar Polygons may be divided into an equal number of triangles, and are in the duplicate proportion of their homologous sides.



IF the Polygons $ABCDE$ and $GHLM$ be similar, they may be divided into an equal number of similar triangles.

which will be the similar parts of their wholes.
Draw the lines AC, AD, GI, GL .

Demonstration.

Since the Polygons are similar, their angles B and H will be equal; and AB will have the same proportion to BC as GH to HI , (by definition 1.) therefore the triangles ABC and GHI are similar, (by the 6.) and (by the 4.) BC has the same proportion to CA as HI to GI . Furthermore because CD has the same proportion to BC as IL to IH , and BC the same to CA as HI to GI ; by equality, CD will have the same proportion to CA as IL to GI . Now the angles BCA

BCD and HIL being equal, if the angles ACB and GIH , which are equal, be taken from them, the angles ACD and GIL will remain equal. Therefore (*by the 6.*) the triangles ACD and GIL will be similar. In like manner, 'tis easie to run over all the triangles of the Polygons, and to prove them similar.

I add further, that the triangles are in the same proportion as the polygons.

Demonstration.

Since all the triangles are similar, their sides will be proportional, (*by the 4.*) but each triangle to its similar is in the duplicate proportion of the homologous sides, (*by the 19.*) therefore every triangle of one polygon to every triangle of the other is in the duplicate proportion of its sides; which being the same, the duplicate proportion must be the same; and there will be the same proportion of each triangle to its similar, as of all the triangles of one polygon to all the triangles of the other polygon, (*by the 12. 5.*) *i.e.* of one polygon to the other.

Coroll. 1. Similar polygons are in the duplicate proportion of their sides.

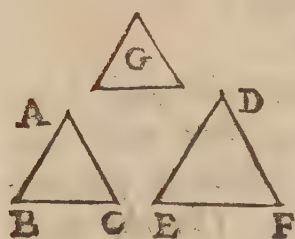
Coroll. 2. If three lines be in continual proportion, a polygon describ'd upon the first will have the same proportion to a polygon describ'd upon the second, as the first line to the third, *i.e.* it will be in the duplicate proportion

tion of that of the first line to the second.

PROPOSITION XXI.

A THEOREM.

Polygons, that are similar to another polygon, are so also amongst themselves.



IF two polygons be similar to a third, they will be so also betwixt themselves. For they may each be divided into as many similar triangles, as are in the third.

But triangles similar to a third, are also similar amongst themselves; because angles equal to one another, are equal amongst themselves; and the angles of the triangles being equal, those of the polygons being compounded of them must be likewise.

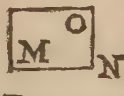
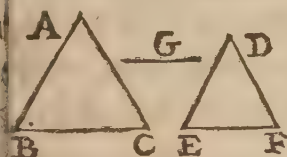
I add, that the sides of the triangles being proportional, those of the polygons must be also, because they are the same.

PRO

PROPOSITION XXII.

A THEOREM.

Similar polygons describ'd upon four proportional lines, are also proportional. And if the polygons be proportional, the lines are so too.



P—

IF BC has the same proportion to EF as HI to MN, the polygon ABC will also have the same proportion to the similar polygon DEF, as HL to its similar polygon MO. Seek

third proportional G to the lines BC and EF, and to the lines HI and MO another third proportional P, (*by the 11.*) Since BC has the same proportion to EF as HI to MN, and EF to G as MN to P; by equality, BC will have the same proportion to G, as HI to P: and this proportion will be the double of that of BC to EF, or HI to MN.

Demonstration.

The polygon ABC to the polygon DEF is in the duplicate proportion of that of BC to EF, (*by the 20.*) that is, as BC to G; and the polygon HL has the same proportion to MO, as HI to P. Therefore ABC has the same pro-

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portion

portion to DEF, as HL to MO.

And if the similar polygons be proportional the lines being in the subduplicate proportion to them, will be also proportional.

The U S E.

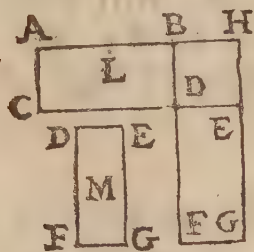
A, B ; C, D,
3, 2 ; 6, 4,
9, 4 ; 36, 16,
F, F ; G, H,

This proposition may be easily apply'd to numbers, if the numbers A, B, C, D, be proportional, their squares E, F, G, H, will be so also; which is very serviceable in *Arithmetick*, and more in *Algebra*.

PROPOSITION XXIII.

A THEOREM.

Equiangular Parallelograms are in the proportion compounded of the proportions of their sides.



IF the parallelograms L and M be equiangular the proportion of L to M will be compounded of the proportion of AB to DE, and that of BD to DF. Joyn the parallelograms so, that the sides BD and DF may make but one right line as also CD and DE another; which, the

parallelograms being equiangular, may be done, [by the coroll. of the 15. 1.] and compleat the parallelogram BDEH.

Demonstration.

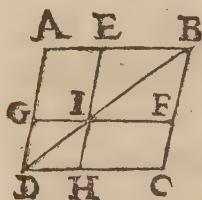
The parallelogram L has the same proportion to the parallelogram BDEH, as the base AB to the base BH or DE, [by the 1.] and the parallelogram BDEH has the same proportion to the parallelogram DFGE, i.e. M, as the base BD to DF. But the proportion of the parallelogram L to the parallelogram M is compounded of that of L to the parallelogram BDEH, and of that of BDEH to the parallelogram M. Therefore the proportion of L to M is compounded of that of AB to DE, and that of BD to DF. For example, let AB be 8, DE 5, BD 4, DF 7; and make as 4 to 7, so 5 to $8\frac{3}{4}$; by which means you will have three numbers, 8, 5, and $8\frac{3}{4}$; 8 to 5 being the proportion of the parallelogram L to BDEH, which is that of AB to DE; and 5 to $8\frac{3}{4}$ that of the parallelogram BDEH to M. Taking away therefore the middle term five, there will remain 8 to $8\frac{3}{4}$ for the proportion compounded of the two.

PROP.

PROPOSITION XXIV.

A THEOREM.

In all parallelograms, those through which the diameter passes, are similar to the great one.



Suppose the diameter of the parallelogram AC pass through the parallelograms EF, GH: I say they are similar to the parallelogram AC.

Demonstration.

The parallelograms AC and EF have the same angle B: and because in the triangle BCD, IF is parallel to the base DC, the triangles BFI and BCD are equiangular. Therefore [by the fourth] BC has the same proportion to CD as BF to FI, and consequently the sides are in the same proportion. In like manner IH being parallel to BC, DH will have the same proportion to HI as DC to BC; the angles are also equal, all the sides being parallels: therefore [by defin. 1.] the parallelograms EF and GH are similar to the parallelogram AC.

The U S E.

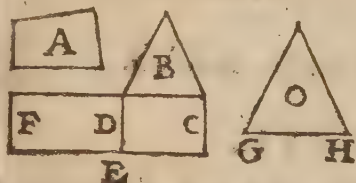
I have made use of this proposition to demonstrate the 10. proposition of my last book

of *Perspectives*, where I have shown a way to draw an Image similar to the Original, by a parallelogram compos'd of four Rules.

PROPOSITION XXV.

A PROBLEM.

To describe a polygon similar to one polygon given
and equal to another.



IF you desire to describe a polygon equal to the rectilineal A, and similar to the polygon B, make

a parallelogram CE equal to the polygon B, [by the 44.1.] and upon the line DE make another parallelogram equal to the rectilineal A, [by the 45.1.] Then find a middle proportional GH between CD and DF, [by the 13.] Lastly, make upon GH a polygon O, similar to B, [by the 18.] which will be equal to the rectilineal A.

Demonstration.

Since CD , GH , and DF , are in a continual proportion, the rectilinear B describ'd upon the first, will have the same proportion to the rectilinear O describ'd upon the second, as CD to DF , [by coroll. 2. of the 20.] But as CD to DF , so is the parallelogram CE to FE , or B to A , which

which are equal to them. Therefore B has the same proportion to O as B to A, and consequently [by the 9.5.] A and O are equal.

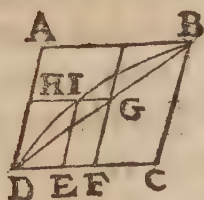
The U S E.

'This proposition teaches how to change one figure for another, retaining still its equality to a third; which is very useful in *practical Geometry*, for the reducing all figures to squares.

PROPOSITION XXVI.

A THEOREM.

If in one angle of a parallelogram you describe a less, similar to the former, the diameter of the greater will fall upon the angle of the less.



IF in the angle D of the parallelogram AC you describe a lesser parallelogram DG, similar to the other, the diameter BG will pass by the point G. For if it do not pass by that point, suppose it then to pass by the point I, and to make the line BID. Draw the line IE parallel to HD.

Demonstration.

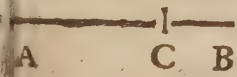
The parallelogram DI would be similar

the parallelogram AC, (*by the 24.*) But the parallelogram DG is also suppos'd similar to it, therefore the parallelograms DI and DG would be similar, which is impossible; for so HI would have the same proportion to IE or GF, as HO to the same GF; and (*by the 9. 5.*) the lines HI and HG would be equal.

PROPOSITION XXX.

A PROBLEM.

To divide a line according to the extreme and middle proportion.

 LET AB be the line propos'd to be divided according to the extreme and middle proportion, *i. e.* so, that AB may have the same proportion to AC as AC to CB. Divide the line AB (*by the 11. 2.*) so, that the rectangle contain'd under AB and CB may be equal to the square of AC.

Demonstration.

Since the rectangle under AB and CB is equal to the square of AC, AB will have the same proportion to AC as AC to CB, (*by the 17.*)

The USE.

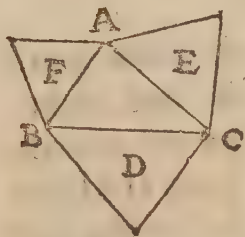
'This proposition is necessary in the Thirteenth book of *Euclid*, for the finding the side
P p of

of five regular bodies. And Frier *Lucas*, of the *Holy Sepulcher*, has compos'd a whole book concerning the properties of a line divided according to the extreme and middle proportion.

PROPOSITION XXXI.

A THEOREM.

A polygon describ'd upon the base of a rectangular triangle, is equal to the two similar polygons describ'd upon the other sides of the same triangle.



IF the angle BAC of the triangle ABC be a right angle, the polygon D, describ'd upon its base BC, will be equal to the two similar polygons F and E describ'd upon the sides AB and AC.

Demonstration.

The polygons D, E, and F, are amongst themselves in the duplicate proportion of their homologous sides BC, AC, and AB, (*by the 20. 1.*) and if squares were describ'd upon the same sides, they also would amongst themselves be in the duplicate proportion of their sides; but [*by the 47. 1.*] the square of BC would be equal to the sum of the squares of AB and AC.

the squares of AC and AB : therefore the polygon D describ'd upon the base BC, will be equal to the similar polygons E, and F, describ'd upon AB and AC.

The USE.

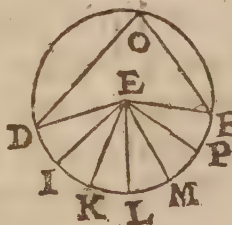
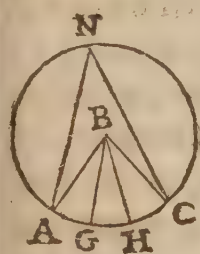
'This proposition is made use of to augment or diminish all manner of figures, being more universal than the 47.1. which yet is exceeding useful, in as much as almost all Geometry is grounded upon that principle.

The 32. Proposition is useless.

PROPOSITION XXXIII.

A THEOREM.

In equal circles, the angles as well at the center as circumference, as also the sector, are in the same proportion as the arches upon which they stand.



IF the circles ANC and DOF are equal, the angle ABC will have the same proportion to the

angle DEF as the arch AC to the arch DF. Suppose AG, GH, and HC, to be equal arches,

Pp 2

and

and consequently the aliquot parts of AC ; and let DF be divided into as many parts, equal to AG , as it contains; and draw the lines EL, EK and the rest.

Demonstration.

All the angles, ABG, GBH, HBC, DEI, IEK , and the rest, are equal, (*by the 27.3.*) so that AG , an aliquot part of the arch AC , will be contain'd in the arch DF , as oft as the angle ABG , an aliquot part of the angle ABC , is contain'd in the angle DEF ; therefore the arch AC will have the same proportion to the arch DF , as the angle ABC to the angle DEF . And because N and O are the halves of the angles ABC and DEF , they will be in the same proportion as these: therefore the angle N has the same proportion to the angle O , as the arch AC to the arch DF .

The same holds likewise of the Sectors: for if you draw the lines AG, GH, HC, DI, IK , and the rest, they will be equal, (*by the 29.3.*) and each little sector will be divided into a triangle and a segment. But the triangles will be equal (*by the 8.1.*) and the little segments will also be equal, (*by the 24.3.*) therefore the whole little Sectors will be equal; and consequently as many aliquot parts of the arch AC as are contain'd in the arch DF , so many aliquot parts of the sector ABC will be contain'd in the sector DEF . Therefore the arch has the same proportion to the arch, as the sector to the sector.

THE

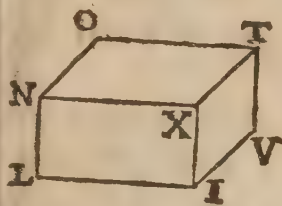
THE ELEVENTH BOOK
OF THE
ELEMENTS
OF
EUCLID.

THis Book establishes the first principles relating to solid bodies ; insomuch that it is impossible to know any thing certainly concerning the third *species* of quantity, without understanding what is herein taught. Upon which account the knowledge of it is absolutely necessary to a through-insight into the greatest part of *Mathematical* Treatises. In the first place, the Doctrine of the Sphere deliver'd by *Theodosius* does suppose a perfect knowledge of the whole. In like manner *Spherical Trigonometry*, the third part of *Practical Geometry*, divers propositions of *Statics* and *Geography* are built upon the principles of Solids. The main difficulties in *Gnomonicks*, *Conick Sections*,

'tions, and the Tracts concerning the cutting of
'pretious Stones, arising chiefly from their emi-
'nencies and rais'd parts, not easily represented
'upon paper, and their being contain'd under
'many superficies's, are render'd intelligible
'and easie by the previous knowledge of the
'doctrine of Solids.

'I have omitted the seventh, eighth, ninth,
'and tenth Books of the *Elements* of *Euclid*, be-
'ing of little or no use in any part of the *Ma-*
'*thematicks*. And I have oft wondred how they
'obtain'd a place amongst the *Elements*, since the
'evident *Euclid* compil'd them for no other end
'but to settle the doctrine of Incommensurables;
'bles; which being little better than a vain
'curiosity, ought not to be receiv'd into the
'books which treat of the *First Principles* of the
'Science, but to make a particular Treatise by it-
'self. The same may be said of the Thirteenth
'book, and those that follow it. And therefore
'tis my opinion, that almost all parts of the *Ma-*
'*thematicks* may sufficiently be understood by
'the help of these Eight books of the *Elements*
'of *Euclid*.

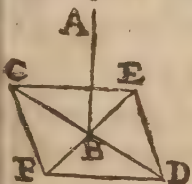
DEFINITIONS.



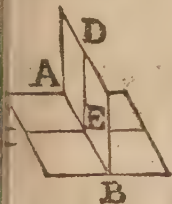
1. **A** Solid body is a quantity, that hath length, bredth, and depth,, or thickness. "As the figure LT, whose

"length is NX, bredth NO, and thickness "LN.

2. The extremes or terms of a solid body are superficies's.



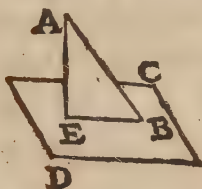
3. A Line is right, or perpendicular to a plane, when tis perpendicular to all the lines, which it meets in the plane. "As the line AB will be right to the plane CD, if it be perpendicular to the lines CD and FE, which being drawn upon the plane CD, pass by the point B, so that the angles ABC, ABD, ABE, and ABF, are right angles.



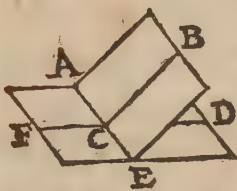
4. One plane is perpendicular to another, when a perpendicular line drawn upon one of them to the common section, is also perpendicular to the other.

" We call the line that is common

“mon to both the planes the common section
 “of the planes: As the line AB, which is ass
 “well in the plane AC, as in the other AD. If
 “therefore the line DE, drawn on the plane
 “AD, perpendicular to AB, be also perpendi-
 “cular to the plane AC, the plane AD will be
 “right to the plane AC.



5. If the line AB be not per-
 pendicular to the plane CD, and
 from the point A a perpendicular
 be drawn to it AE, and also the
 line BE; the angle ABE is the
 Inclination of the line AB to the plane CD.



6. The Inclination of one
 plane to another, is the acute
 angle form'd by the two per-
 pendiculars drawn upon each
 plane to their common section
 on. “As the Inclination of the plane AB to
 “the plane AD, is nothing else but the angle
 “BCD, form'd by the lines BC and CD, drawn
 “upon the two planes, perpendicular to their
 “common section AE.

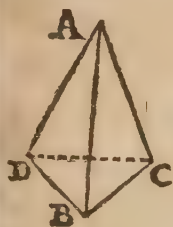
7. Planes are inclin'd after the same manne
 if their angles of Inclination be equal.

8. Planes are parallel, if being continu'd
 far as you please, they still retain the same di-
 stance one from the other.

9. Solid figures are similar, which are con-
 tain'd within, or terminated by, an equal num

ber of similar planes; as two Cubes. "This
"definition does not agree to those figures,
"whose superficies's are crooked; as the Sphere,
"the Cylinder, and the Cone.

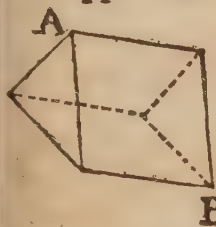
10. Equal and similar solid figures are con-
tain'd within, or terminated by, an equal num-
ber of equal and similar planes. " Infomuch,
"that if they were suppos'd to penetrate each
"other, neither of them would exceed, having
"their sides and angles equal.



11. A solid angle is the con-
course, or inclination, of divers
lines, in different planes. " As the
"concourse of the lines AB, AC,
"and AD, which are in different
planes.

12. A Pyramid is a solid figure, terminated
by triangles, whose bases are in the same plane.
"As the figure ABCD.

13. A Parallelepipedon is a solid figure con-
tain'd within six quadrilateral planes, of which
the opposites are parallel.



14. A Prisme is a solid fi-
gure, having two parallel planes
similar and equal, and the o-
thers Parallelograms. "As the
"figure AB. Its opposite planes
"may be polygons.

15. A Sphere is a solid figure, terminated
by one only superficies; from which divers
lines

lines being drawn to a point in the middle of the figure, they will be all equal. "Some define a Sphere by the motion of a semicircle, "turn'd about upon its diameter, which remains immoveable.

16. The Axis of a Sphere is that immovable line about which the semicircle is turn'd.

17. The Center of the Sphere, is the same with that of the semicircle, by whose motion it is made.

18. The Diameter of a Sphere, is any line whatsoever passing through its center, and terminated at the superficies.



19. If a line, immoveable at one of its points, taken above the plane of a circle, be mov'd about the circumference, it will describe a Cone. "As if the line AB, being fix'd at the point A, be mov'd about the circumference BED, it will describe the Cone ABED. "The point A will be its summity or vertex, "and the circle BED its base.

20. The Axis of a Cone, is the line drawn from the vertex to the center of the base. "As AC.



21. If a line be mov'd about the circumference of two parallel circles, so that it remains always parallel to a line drawn from the center of one of the circles to that of the other,

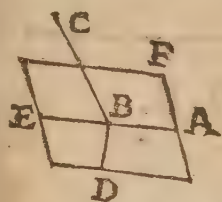
ther, i. e. the Axis, it will describe a Cylinder.

22. Cones are said to be right, when the axis is perpendicular to the plane of the base. Also right cones are similar, when their axis's and the diameters of their bases are in the same proportion. But Inclined cones are not similar, unless they have a third condition; that their axis's be equally inclin'd to the planes of their bases.

PROPOSITION I.

A THEOREM.

A right line cannot have one of its parts upon a plane, and the other above or below it.



IF the line AB be upon the plane AD, it will not, being continu'd, either rise above or fall below it, but all its parts will lie upon the same. For if

it be possible that BC can be a part of AB continued, draw upon the same plane AD the line BD perpendicular to AB, and also BE perpendicular to BD upon the same.

Demonstration.

The angles ABD and DBE are two right angles; therefore (by the 14. 1.) AB and BE make but one right line, and consequently BC

Qq 2

is

is no part of the line AB continued : otherwise two right lines CB and EB would have the same part AB in common , which is repugnant to the 13. Axiom of the first book.

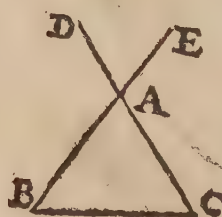
The USE.

Upon this proposition is built a principle in *Gnomonicks*, by which we prove, that the *shadow* of the *Style* cannot fall out of the plane of a great circle, in which is the Sun. For the extremity of the *Style* being taken for the centre of the heavens, and consequently of all the greater circles, and the *shadow* being always in a right line of a Ray drawn from the Sun to the opacous body, and this ray being in the plane of this great circle, the *shadow* must be so likewise.

PROPOSITION II.

A THEOREM.

Lines that cut each other, are in the same plane, as are also all the parts of a triangle.



IF the two lines BE and CD cut each other at the point A , and a triangle be form'd by drawing the base BC ; I say, all the parts of the triangle ABC

are in the same plane, and also the lines BE and CD.

Demonstration.

It cannot be said that any part of the triangle ABC is in a plane, and another part of the same triangle not in the same, but it must be also affirm'd, that one part of a right line is in a plane, and another part of the same line is not in the same plane; which is contrary to *Prop. 1.* And because the sides of the triangle must be in the same plane in which is the triangle, the lines BE and CD will be also in the same plane.

The U S E.

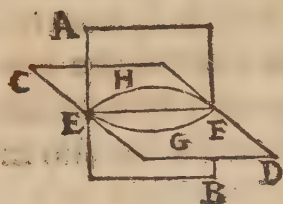
'This proposition sufficiently determines a plane, by the concourse of two right lines, or by a triangle. I have also made use of it in *Opticks*, to prove that objective parallel lines, which meet upon a Table, ought to be represented by lines that concur in a point.

PROP.

PROPOSITION III.

A THEOREM.

The common section of two planes is one right line.



IF the planes AB and CD cut each other, their common section EF will be one right line. For if not, take two points common to both planes, as E and F; and draw a right line from the point E to the point F upon the plane AB, which suppose to be EHIF. Draw likewise upon the plane CD a right line from the same point E to F; and if it be not the same with the former, suppose it to be EGIF.

Demonstration.

These lines drawn upon two planes are two different lines, and enclose space; which is contrary to the 12. *Axiom.* of the 1. Therefore they will make but one right line, which being in both the planes will be their common section.

The USE.

‘This is a fundamental proposition, suppos’d
‘divers parts of the *Mathematicks*, though
‘be not always quoted. Particularly, it is t
‘ken for granted in *Gnomonicks*, when the hou

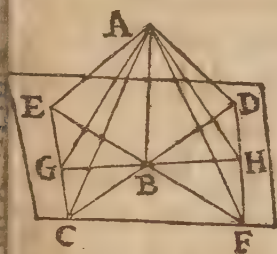
lin

lines are represented upon *Dials*, by marking only the common section of their plane, and that of the wall.

PROPOSITION IV.

A THEOREM.

If a line be perpendicular to two others that cut each other, it will also be perpendicular to the plane of the same lines.



IF the line AB be perpendicular to the lines CD and EF, which cut each other at the point B, so that the angles ABC, ABD, ABE, and ABF, be right angles, (which

cannot conveniently be represented upon a plate,) it will be also perpendicular to the plane of the lines CD and EF, *i.e.* to all the lines that shall be drawn upon the same plane through the point B; as, for example, the line GBH. Let the lines BC, BD, BE, and BF, be equal, and draw the lines EC, DF, AC, AD, AE, AF, AG, and AH.

Demonstration.

The four triangles ABC, ABD, ABE, and ABF, have each a right angle at the point B; and the sides BC, BD, BE, and BF equal, with the

the side AB common to all. Therefore their bases AC, AD, AE , and AF , are equal, (*by the 4. 1.*)

2. The triangles EBC and DBF will be in all respects equal, having their sides BC, BD, BE , and BF , equal; and the angles CBE and DBF , being oppos'd at the top, equal: therefore the angles BCE, BDF, BEC , and BFD will be equal, (*by the 4. 1.*) and also the bases EC and DF .

3. The triangles GBC , and DBH , having the opposite angles CBG , and DBH , equal; and also the angles BDH , and BCG ; and the sides BC and BD ; the sides BG and BH, CG and DH , will be also equal, (*by the 26. 1.*)

4. The triangles ACE and AFD , having their sides AC, AD, AE , and AF , equal, and the bases EC and DF also equal; the angles ADF and ACE will be equal, (*by the 8. 1.*)

5. The triangles ACG and ADH have their sides AC and AD, CG and DH equal, with the angles ADH and ACG ; therefore their bases AG and AH are equal.

Lastly, the triangles ABH and ABG have all their sides equal; therefore (*by the 8. 1.*) the angles ABG and ABH will be equal, and the line AB perpendicular to GH . Accordingly this line AB will be perpendicular to any line drawn through the point B upon the plane of the lines CD and EF , which I call being perpendicular to their plane.

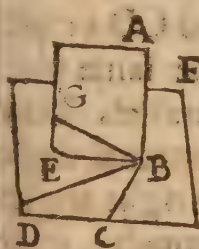
The USE.

'This Proposition occurs very oft in the first book of *Theodosius*: for example, to demonstrate that the *Axis* of the world is perpendicular to the plane of the *Equinoctial*. In like manner in *Gnomonicks*, tis demonstrated by this proposition, that the *Equinoctial* line in *Horizontal Dials* is perpendicular to the *Meridian*. Nor is it less useful in other *Mathematical Treatises*; as those concerning *Astro-labes*, and the cutting of precious stones.

PROPOSITION V.

A THEOREM.

If a line be perpendicular to three others, which cut each other at the same point, they will be all three in the same plane.



IF the line AB be perpendicular to three lines BC, BD, and BE, which cut each other at the same point B, the lines BC, BD, and BE, are in the same plane. Suppose the plane AE to be that of the lines AB and BE, and CF that of the lines BC and BD. If BE be the common section of both the planes, it will be in the plane

R r

of the lines BC and BD , as was asserted : but BE be not, let BG be their common section.

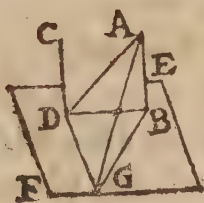
Demonstration.

AB is perpendicular to the lines BC and BD therefore it is perpendicular to their plane CBD (by the 4.) and (by defin. 3.) AB will be perpendicular to BG . But it is also suppos'd perpendicular to BE ; therefore the angles ABG and ABE are right angles, and consequently equal, though one be part of the other. Therefore the two planes can have no other common section but BE , BE is therefore in the plane CF .

PROPOSITION VI,

A THEOREM.

Lines that are perpendicular to the same plane, are parallel.



IF the lines AB and CD be perpendicular to the same plane EF , they will be parallel. This is evident, that the internal angles ABD and BDC are right angles ; but that is not enough ; it remains to be prov'd, that AB and CD are in the same plane. Draw DG perpendicular to AB and equal to AB ; draw also the lines BG , AD .

Demonstration.

The triangles ABD and BDG have the sides AB and DG equal, and BD common to both : and the angles ABD and BDG are right angles, therefore their bases AD and BG are equal, (*by the 4. 1.*) Further, the triangles ABG and ADG have all their sides equal : therefore the angles ABG and ADG are equal ; and ABG being a right angle, because AB is perpendicular to the plane, ADG is also a right angle. Therefore the line DG is perpendicular to three lines CD, DA, and DB, which consequently are in the same plane, (*by the 5.*) but the line AB is in the plane of the lines AD and DB, (*by the 2.*) therefore AB and CD are in the same plane.

Coroll. Two parallel lines are in the same plane.

The U S E.

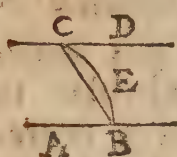
'By this proposition we demonstrate, that the 'hour-lines, in all planes that are parallel to 'the *Axis* of the world, as the Polars, Meridional, and others, are parallel among themselves.

PROP.

PROPOSITION VII.

A THEOREM.

A line drawn from one parallel to another, is in the same plane with them.



THE line CB, being drawn from the point B of the line AB to the point C of its parallel CD is (I say) in the plane of the lines AB and CD.

Demonstration.

The parallels AB and CD are in the same plane : in which if you draw a right line from the point B to the point C, it will be the same with CB ; otherwise two right lines would enclose space, contrary to the 12. *Axiom.* of the

PROPOSITION VIII.

A THEOREM

If one of two parallel lines be perpendicular to a plane, the other will be so also.

IF of the two parallel lines AB and CD, [see prop. 6.] the one AB be perpendicular to a plane

plane EF, the other CD will be so also. Draw the line DB : since the angle ABD is a right angle, and AB and CD are suppos'd to be parallels, the angle CDB will be a right angle, (*by the 29. 1.*) therefore if I can prove, that the angle CDG is also a right angle, it will follow (*by the 4.*) that CD is perpendicular to the plane EF. Make a right angle BDG, and take DG equal to AB ; then draw the lines BG and AG.

Demonstration.

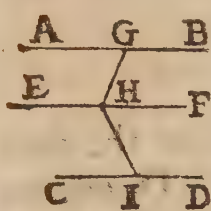
The triangles ABD and BDG have the sides AB and DG equal, with the side BD common to both ; and the angles ABD and BDG are right angles : therefore (*by the 4. 1.*) their bases AD and BG are equal. The triangles ADG and ABG have all their sides equal, therefore (*by the 8. 1.*) the angles ADG and ABG are equal : But the latter is a right angle, because AB is suppos'd to be perpendicular to the plane EF, therefore the angle ADG is a right angle ; and the line DG being perpendicular to the lines DB and DA, will be perpendicular to the plane of the lines DB and DA, which is the same in which are the parallels AB and CD. Therefore the angle GDC is a right angle, (*by defin. 3.*)

PROP.

PROPOSITION IX.

A THEOREM.

Lines, that are parallel to a third, are also parallel among themselves, though not all in the same plane.



IF the lines AB and CD are parallel to the line EF , they will be parallel to each other, though all the three lines be not in the same plane. Upon the plane of the lines AB and EF draw the line EH perpendicular to AB ; which will be also perpendicular to EF , [by the Lemma after the 26.1.] In like manner upon the plane of the lines EF and CD draw the line HI perpendicular to EF and CD .

Demonstration.

The line EH being perpendicular to the lines GH and HI , is so also to the planes of the lines HG and HI , (by the 4.) therefore (by the 8.) the lines AG and CI are perpendicular to the plane of the lines HG and HI , and [by the 6.] parallel to each other.

The USE.

'This proposition is frequently used in *Perspectives*, to determine the representation of 'parallel

'parallel lines upon a table ; as also in the cutting of pretious Stones, to prove the sides of the Pannels to be parallel among themselves, because they are so to a line in a different plane. In *Gnomonicks* likewise we are sometimes oblig'd to prove, that the Vertical circles ought to be describ'd on walls by perpendicular lines ; because the lines, that are the common sections of them and the walls, are parallel to a line drawn from the *Zenith* to the *Nadir*.

PROPOSITION X.

A THEOREM.

If two lines, which concur, are parallel to two others concurring, of a different plane, they will make equal angles.



IF the lines AB and CD, AE and CF be parallel, though they be not all four upon the same plane, yet the angles BAE and DCF will be equal. Let the lines AB and CD, AE and CF be equal, and draw the lines BE, DF, AC, BD, and EF.

Demonstration.

The lines AB and CD are suppos'd to be both parallel and equal, therefore [by the 33.1.] the

the lines AC and BD are parallel and equal, as also AC and EF; and [by the preceding] BD and EF are parallel, and equal, and consequently [by the 33. 1.] BE and DF will be also parallel and equal. Therefore the triangles BAE and DCF have all their sides equal: and [by the 8. 1.] the angles BAE and DCF will be equal.

Coroll. Many the like propositions might be made, which would not be altogether unuseful; as for example, if upon a parallel plane the lines CD be drawn parallel to the line AB, and the angles BAE and DCF be equal, the lines AE and CF will be parallel.

The U S E.

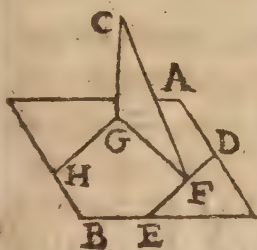
‘By this proposition we demonstrate, that the angles made by the planes of the hour-circles with a plane parallel to the Equator, are equal to the angles made by them with the plane of the Equator.

PROP

PROPOSITION XI.

A PROBLEM.

To draw a perpendicular to a plane from a point given out of the plane.



IF you desire to draw a perpendicular from the point C to the plane AB, draw the line EF at pleasure, and CF perpendicular to it, [by the 12. 1.]

And again [by the 11. 1.] upon the plane AB draw FG perpendicular to ED, and CG perpendicular to FG. I say, CG will be perpendicular to the plane AB. Draw GH parallel to FE.

Demonstration.

The line EF being perpendicular to the lines CF and FG, will be perpendicular to the plane CFG, [by the 4.] and HG being parallel to EF, will be also perpendicular to the same plane, [by the 8.] And because CG is perpendicular to the lines GF and GH, it will be perpendicular to the plane AB, [by the 4.]

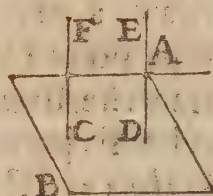
S. S.iffel

PROP.

PROPOSITION XII.

A PROBLEM.

To draw a perpendicular to a plane through a point of the same plane.



TO draw a perpendicular to the plane AB through the point C, draw from the point E, taken at pleasure out of the plane, the line ED perpendicular to the same plane, [by the 11.] Draw also [by the 31. 1.] CF parallel to DE. CF will be perpendicular to the plane AB, [by the 8.]

PROPOSITION XIII.

A THEOREM.

Two lines perpendicular to a plane cannot be drawn through the same point.



IF the two lines CE and CD drawn through the same point C, were perpendicular to the plane AB, and CF the common section of the planes of those lines, with the plane AB; the angles ECF and

DCE

DCF would be both right angles, which is impossible.

I adde, that two perpendiculars DC and DF to the plane AF cannot be drawn from the same point D: for having drawn the line CF, there would be two right angles, DCF and DFC, in the same triangle, *contrary to the 32. 1.*

The U. S. E.

'This proposition is necessary to show, that a perpendicular to a plane was sufficiently described, in as much as but one such can be drawn through the same point.

PROPOSITION XIV.

A THEOREM.

Planes, to which the same line is perpendicular, are parallel.



IF the line AB be perpendicular to the planes AC and BD, they will be parallel, *i. e.* they will in all places be equally distant from each other. Draw the line DC parallel to AB, [*by the 31. 1.*] and joyn the lines BD and AC.

Demonstration.

AB is suppos'd to be perpendicular to the
S S 2 planes

planes AC and BD, therefore the line CD which is parallel to it, will be also perpendicular to them, [*by the 8.*] and consequently the angles B and D, A and C, will be right angles and [*by the 28. 1.*] the lines AC and BD will be parallels, and the figure ABDC a parallelogram. Therefore the lines AB and CD are equal, [*by the 34. 1.*] i.e. the planes in the points A and C, B and D; are equally distant. Accordingly the line CD may be drawn through any other point whatsoever; therefore the planes AC and BD are equally distant in all places the one from the other.

The U S E.

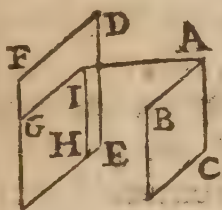
'Theodosius demonstrates the circles, that have the same poles, as the *Equator*, and the two *Tropicks*, to be parallel, because the *Axis* of the world is perpendicular to their planes.

PRO

PROPOSITION XV.

A THEOREM.

If two lines, meeting at a point, be parallel to two lines of another plane, the planes of those lines will be parallel.



IF the lines AB and AC be parallel to the lines DE and DF, which are in another plane, the planes BC and FE are parallel. Draw AI perpendicular to the plane BC, [by the 11.] and GI and IH parallel to FD and DE: they will be also parallel to the lines AB and AC, [by the 9.]

Demonstration.

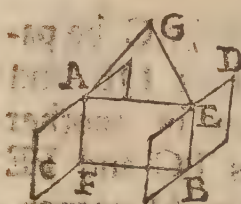
The lines AB and GI are parallel, and the angle IAB is a right angle, AI being perpendicular to the plane BC: therefore [by the 29.1.] the angle AIG is a right angle, as also the angle AIH. Therefore [by the 4.] the line AI is perpendicular to the plane GH; and being also perpendicular to the plane BC, the planes BC and GH, or FE, will be parallel, [by the 14.]

PROP.

PROPOSITION XVI.

A THEOREM.

If a plane cut two others which are parallel, their common sections will together with them be parallel.



IF the plane AB cut two other parallel planes, AC and BD ; I say, their common sections AF and BE will be parallel. For if not, being continu'd they would at length concur, e. g. at the point G.

Demonstration.

The lines AF and BE are upon the planes AC and BD ; and therefore [by the 1.] can never be either above, or below it ; therefore if they concur at the point G, the planes must do so likewise, and consequently they would not be parallel, which is contrary to what was suppos'd.

The USE.

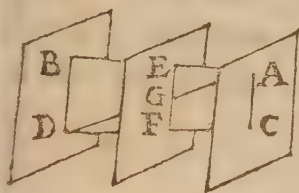
By this Proposition we demonstrate, in the *Treatise of Conick and Cylicdrick Sections*, that if the *Cone* or *Cylinder* be cut by a plane parallel to its base, the sections are circular. By the same we describe *Astrolabes* ; and prove in *Gnomonicks*, that the angles, which the

the hour circles make with a plane parallel to a great circle, are equal to those which they make in the circle it self; and again in *Perspectives*, that the Images of the objective lines perpendicular to the table, concur at the point of sight.

PROPOSITION XVII.

A THEOREM.

Two lines are divided proportionally by parallel planes.



IF the lines AB and CD be divided by parallel planes, I say, AE will have the same proportion to EB as CF to FD. Draw the line

AD, passing through the plane EF at the point G: Draw also AC, BD, FG, and GE.

Demonstration.

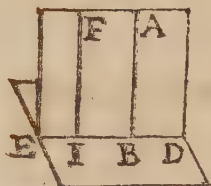
The plane of the triangle ABD cuts the three planes, therefore (*by the 16.*) the sections BD and EG are parallel; and (*by the 2. 6.*) AE has the same proportion to EB, as AG to GD. In like manner the plane of the triangle ADC cuts the planes EF and AC, therefore the sections AC and FG are parallel; and FC has the same proportion to FD as AG to GD, *i.e.* as AE to EB.

PROP.

PROPOSITION XVIII.

A THEOREM.

If a line be perpendicular to a plane, all the planes, in which that line is found, are perpendicular to the same plane.



IF the line AB be perpendicular to the plane ED, all the planes in which it is found will be perpendicular to the plane ED. Suppose AB to be in the plane AE, having for a common section with the plane ED the line BE; to which draw a perpendicular FI.

Demonstration.

The angles ABI and BIF are right angles; therefore the lines AB and FI are parallel; and (by the 8.) FI will be perpendicular to the plane ED. Therefore the plane AE will be perpendicular to the plane ED, (by def. 4.)
 "The same may be prov'd of the plane AD.

The USE.

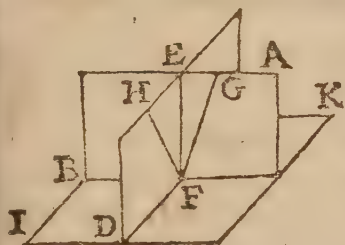
"The first Proposition in *Gnomonicks*, which may pass for a Fundamental one, is built upon this proposition; which is also frequently made use of in *Spherical Trigonometry*, in *Perspective*.

*'spectives, and generally in all those Treatises
'which are oblig'd to consider divers planes.*

PROPOSITION XIX.

A THEOREM.

*If two planes cutting each other be perpendicular to
another, their common section will be perpendi-
cular to the same,*



IF the planes AB and ED, which cut each other, be perpendicular to the plane IK, their common section EF is also perpendicular to the plane IK.

Demonstration.

If EF be not perpendicular to the plane IK, upon the plane AB draw the line GF perpendicular to the common section BF: and the plane AB being perpendicular to the plane IK, the line GF will be perpendicular to the same plane. Draw likewise FH perpendicular to the common section DF; it will be also perpendicular to the plane IK. We shall have therefore two perpendiculars to the same plane, drawn through the same point F, (*contrary to the*

13. Propof.) it muſt therefore be granted that EF is perpendicular to the plane IK.

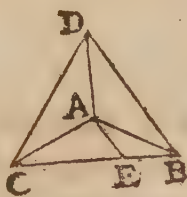
The U S E.

‘By this propoſition we demonſtrate, that the
‘circle which paſſes through the Poles of the
‘world and the *Zenith* is the *Meridian*, and
‘cuts all the diurnal arches into two equal
‘parts ; and that the Stars ſpend as much time
‘in their motions from their riſings to this circle,
‘as from the circle to their ſettings.

PROPOSITION XX.

A THEOREM.

If three plain angles make one ſolid one, any two of them ought to be greater than the third.



IF the angles BAC, BAD, and CAD, make the ſolid angle, and the angle BAC be the greateſt angle ; the two others, taken together, are greater than BAC.

Suppoſe the angle CAE to be equal to the angle CAD, and the lines AD and AE to be equal ; and draw the lines CEB, CD, and BD.

Demonſtration.

The triangles CAE and CAD have the ſide AD and AE equal, and the ſide AC common

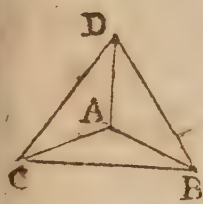
both

both, and the angles CAD and CAE equal. therefore (by the 4. 1.) their bases CD and CE are equal. But the sides CD and DB are greater than the side CB alone, (by the 20. 1.) therefore taking away the equal lines CD and CE, the line BD will be greater than BE. Further, the triangles BAE and BAD have the sides AE and AD equal, and the side BA common, and the base BD greater than the base BE : therefore (by the 18. 1.) the angle BAD is greater than the angle BAE ; adding therefore the equal angles CAD and CAE, the angles BAD and CAD will be greater than the angles CAE and BAE, *i. e.* the angle BAC.

PROPOSITION XXI.

A THEOREM.

All the plain angles, that make one solid angle, are less than four right angles.



IF the plain angles BAC, BAD, and CAD, make the solid angle A, they will be less than four right angles. Draw the lines BC, BD, and CD, and you will have a pyramid, whose base is the triangle BCD.

Demonstration.

The solid angle at the point B, has the angles ABC and ABD greater than that of the

base

base

base alone CBD . In like manner ACB and ACD are greater than BCD alone, and the angles ADC and ADB are greater than CDB alone. But all the angles of the base are equal to two right angles, therefore the angles ABD , ACB , ACD , ADC , and ADB , are greater than two right angles. And because all the angles of the three triangles BAC , BAD , and CAD , are equal to six right angles; taking away more than two right angles, there will remain less than four, for the angles made at the point A . But if the solid angle A consist of more than three plain angles, so that the base of the pyramid be a polygon, it may be divided into triangles; and the computation being made, you will find, that all the plain angles, which make up the solid one, are always less than four right angles.

The use.

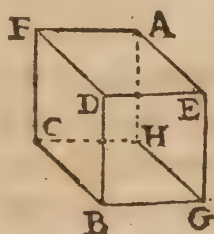
‘These two propositions show when many plain angles may make up one solid one, which is often necessary in the treatises of cutting of Stones, and in the following propositions.

The 22. and 23. Propositions are of no use.

PROPOSITION XXIV.

A THEOREM.

If a solid body be terminated by parallel planes, the opposite sides will be similar and equal parallelograms.



IF the solid AB be terminated by parallel planes, the opposite superficies's will be similar and equal parallelograms.

Demonstration.

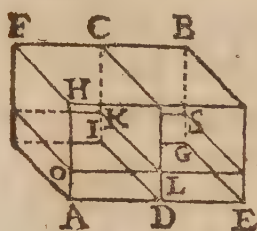
The parallel planes AC and BE are cut by the plane FE: therefore their common sections are parallel, (*by the 16.*) and so likewise DF and AE; therefore AD will be a parallelogram. After the same manner I may demonstrate, that AG, FB, CG, and the rest, are parallelograms. I adde, that the opposite parallelograms, *e. g.* AG and FB, are similar and equal. The lines AE and EG are parallel to the lines FD and DB: therefore the angles AEG and FDB are equal, (*by the 10.*) Accordingly I may demonstrate all the sides and all the angles of the opposite parallelograms to be equal, therefore the parallelograms are similar and equal.

PROP.

PROPOSITION XXV.

A THEOREM.

If a Parallelepipedon be divided by a plane parallel to one of its planes, the two solid bodies which arise by that division, will have the same proportion as their bases.



IF the parallelepipedon ABCE divided by the plane CD, which is parallel to the planes AF and BE, the solid AC will have the same proportion to BD as the base AI to the base DG. Suppose the line

AH, which shows the height of the figure to be divided into as many equal parts as you please; for example, ten thousand; which we may take as indivisibles, *i.e.* without reflecting upon the possibility of their being further subdivided. Suppose also so many superficies's parallel to the base AI, as there are parts in the line AH; I have describ'd only one OS: so that the solid AB be compounded of all those superficies's of the same thickness, as a Ream of paper is compounded of all its sheets and quires laid one upon another. 'Tis evident that so the solid AC will be compounded of ten thousand super-

superficies's equal to the base AI , (*by the preceding,*) and the solid DB will contain ten thousand superficies's equal to the base DG .

Demonstration.

Every superficies of the solid AC has the same proportion to any of the superficies's of the solid DB , as the base AI to the base DG ; because they are every one of them equal to their bases: therefore (*by the 12. 5.*) all the superficies's of the solid AC , taken together, will have the same proportion to all those of the solid DB , as the base AI to the base DG . But all the superficies's of the solid AC make up the solid AC , which has no other parts but those superficies's; and all the superficies's of the solid DB are nothing else but the solid DB ; therefore the solid AC has the same proportion to the solid DB , as the base AI to the base DG .

The USE.

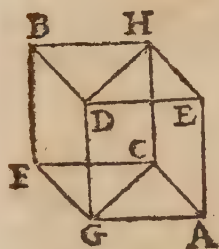
'This is *Cavalieri's* demonstration; which
'is very clear, provided it be used as it ought;
'and that the line, by which is measur'd the
'thickness of the superficies's, be taken in the
'same respect in both the terms. I shall make
'use of it hereafter, to render some intricate and
'perplex'd demonstrations more facil and clear.

PROP.

PROPOSITION XXVI.

A THEOREM.

A parallelepipedon is divided into two equal parts by the diagonal plane.



Suppose the parallelepipedon AB to be divided by the plane CD, drawn from one angle to another: I say it will be divided into two equal parts. Divide the line AE into as many parts as you please, and draw so many planes parallel to the base AF; each of those planes is a parallelogram equal to the base AF, (*by the 24.*)

Demonstration.

All the parallelograms, that can be drawn parallel to the base AF, are divided into two equal parts by the plane CD: for the triangles which are form'd on both sides the plane CD, have their base common, in each equal to CD, and their sides equal, being those of a parallelogram. But 'tis evident, that the parallelepipedon is nothing else but those parallelograms, which are each divided into two equal triangles: therefore the parallelepipedon is divided into two equal parts by the plane CD.

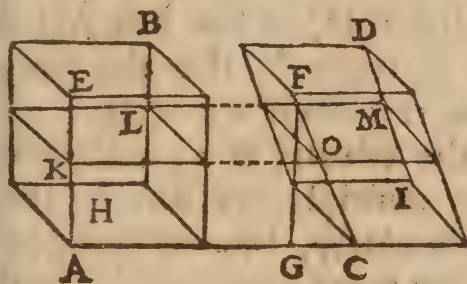
'The

The 27. and 28. Propositions are of no use according to this way of demonstrating.

PROPOS. XXIX, XXX, XXXI.

A THEOREM.

Parallelepipedons of the same hight, having the same or equal bases, are equal.



IF the Parallelepipedons AB and CD be of the same hight according to the perpendiculars AE and FG, and have

the same or equal bases AH and CI, they will be equal. Suppose the two bases to be set upon the same plane; since their perpendiculars are equal, the bases EB and FD will be in the same plane, which will be parallel to the plane of the bases AH and CI. Suppose then the line AE or FG to be divided into as many equal parts as you please, e. g. ten thousand, and according to them so many superficies's or planes drawn of the same thickness: I have describ'd only one for all, as K or M. Each superficies will form in these solids a parallel plane, similar and equal to the base, (*by the 24.*) as KL, OM;
V v and

and there will be as many in one solid as in the other ; because their thickness, which I take perpendicularly according to their respective heights, is equal.

Demonstration.

The base AH has the same proportion to the base CI, as each plane KL to OM. But these being equal in number in both, all the antecedents (*by the 15.5.*) will have the same proportion to all the consequents, *i. e.* the whole solid AB to the whole solid CD, as the base AH to the base CI. But 'tis suppos'd that the bases are equal, therefore the solids are equal.

Coroll. To find the solidity of a parallelepipedon, 'tis usual to multiply the base by the height taken perpendicularly, because that perpendicular shows how many superficies's equal to the base are contain'd in it. As for example, if I take a foot for my indivisible measure, *i. e.* which I will not afterwards subdivide ; if the base contain twelve feet square, and the perpendicular height ten, I shall have an hundred and twenty cubick feet for the solidity of the body AB. For the height containing ten feet, I may make ten parallelograms equal to the base, having each a foot in thickness ; but the base with one foot in thickness makes twelve cubick feet ; the whole therefore will make an hundred and twenty, if the height contain ten feet.

PROP

PROPOSITION XXXII.

A THEOREM.

Parallelepipedons of the same hight are in the same proportion as their bases.

I Have prov'd this proposition in the preceding, demonstrating, that the parallelepipedon AB has the same proportion to the parallelepipedon CD, as the base AH to the base CI. (See fig. preced.)

Coroll. Parallelepipedons that have equal bases, are in the same proportion as their hights. As the parallelepipedons AB and AL, whose perpendicular hights are AK and AE. For if you divide the hight AK into as many aliquot parts as you please, and AE into as many as it contains equal to the former, and draw, according to each part, planes parallel to the base; as many as AE contains of the aliquot parts of AK, so many will the solid AB contain of the superficies's equal to the base, which are the aliquot parts of the solid AL; therefore (by defin. 5. 5.) the solid AB will have the same proportion to the solid AL, as the hight AE to the hight AK.

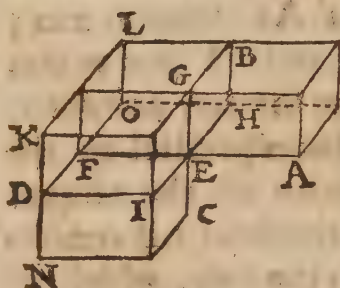
The USE.

'The three preceding propositions contain
'almost all the ways of measuring parallelepipedons,
'pedons, and may be esteem'd as first principles
'for that purpose. Tis after the same manner
'also that we take the dimensions of the solidity
'of Walls, by multiplying their bases by their
'heights.

PROPOSITION XXXIII.

A THEOREM

Similar parallelepipedons are in the triplicate proportion of their homologous sides.



IF the parallelepipedons AB and CD be similar, i.e. if all the planes of one be like those of the other; and all their angles equal, so that they may be placed in a right line, i.e. that AE and EF, HE and EI, GE and EC, may make right lines; and AE has the same proportion to EF as HE to EI, and as GE to EC: I say, that here are four solids in continual proportion according to the proportion of the side EA to that, which is homologous to it, EF or DI.

D

Demonstration.

The parallelepipedon AB has the same proportion to EL of the same height, as the base AH to the base EO, (*by the 32.*) But the base AH has the same proportion to the base EO, as AE to EF, (*by the 1.6.*) In like manner, the proportion of the solid EL to the solid EK, is the same with that of the base EO to the base ED, *i.e.* that of HE to EI. And lastly, the solid EK has the same proportion to the solid EN, as the height GE to the height EC, (*by the coroll. of the 32.*) or (taking the line EF for their common height) as the base GI to the base CI, *i.e.* as GE to EC. But the proportion of AE to EF, of HE to EI, and of GE to EC, was suppos'd to be the same; and consequently, the solid AB has the same proportion to EL as EL to EK, and as EK to CD. Therefore (*by defin. 11. 5.*) the proportion of AB to CD will be the triplicate proportion of that of AB to EL, or of AE to its homologous side EF.

Coroll. If four lines be in continual proportion, the parallelepipedon describ'd upon the first, has the same proportion to another similar parallelepipedon describ'd upon the second, as the first to the fourth; for the proportion of the first to the fourth, is the triplicate proportion of that of the first to the second.

The U S E.

'You may perceive by this proposition that
'that

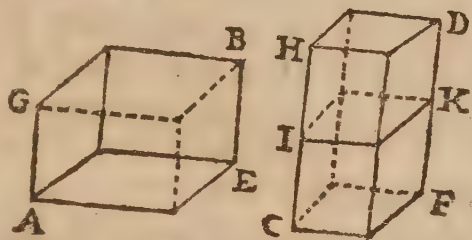
‘that famous problem of the duplication of the
‘Cube, propos’d by the Oracle, consists in find-
‘ing two middle terms in continual proportion.
‘For if you make the side of the first cube the
‘first term, and the double of that the fourth ;
‘having found two middle proportionals, the
‘cube describ’d upon the first line will have the
‘same proportion to that describ’d upon the se-
‘cond, as the first line to the fourth, *i. e.* as one
‘to two. By this proposition also may be cor-
‘rected their error, who fancy similar solids to
‘have the same proportion as their sides ; as if a
‘cube of one foot in length was the half of a
‘cube two foot long ; when indeed it is but the
‘eighth part thereof. This is likewise the foun-
‘dation of the Rule concerning the size of the
‘bores of Canons ; and is applicable not only to
‘bulletts, but to all sorts of similar bodies. For
‘example ; should a man, about to build a Na-
‘vy, and resolving to retain the same proportion
‘in all his vessels, reason thus with himself ; If
‘a Ship of an hundred tun require fifty foot in
‘Keel, another of two hundred tuns ought to
‘have an hundred foot in Keel ; he would be
‘guilty of a great mistake : for instead of mak-
‘ing a vessel twice as large as the former, he
‘would make one eight times so much. He ought
‘to assign to the second vessel somewhat less
‘than sixty three feet.

PROP.

PROPOSITION XXXIV.

A THEOREM.

Equal parallelepipeds have their bases and heights reciprocal, and those that have their bases and heights reciprocal are equal.



IF the parallelepipeds AB and CD be equal, their bases and heights will be reciprocal,

i. e. the base AE will have the same proportion to the base CF, as the height CH to the height AG. Having made CI equal to AG, draw the plane IK parallel to the base CF.

Demonstration.

The parallelepipedon AB has the same proportion to CK, being of the same height, as the base AE to the base CF, (*by the 32.*) But as AB to CK, so is CD to the same CK, because AB and CD are equal; and as CD to CK, which have both the same base, so is the height CH to the height CI, (*by the coroll. of the 32.* therefore as the base AE to the base CF, so is the height CH to the height CI or AG.

I add, that if the base AE has the same proportion to the base CF as the hight CH to the hight AG, the solids AB and CD will be equal.

Demonstration.

AB has the same proportion to CK, being of the same hight, as the base AE to the base CF, (*by the 32.*) Also the hight CH has the same proportion to the hight CI or AG, as CD to CK: but we suppose that AE has the same proportion to CF, as CH to CI or AG; therefore the solid AB has the same proportion to the solid CK as the solid CD to the same CK, and consequently the solids AB and CD are equal, (*by the 9.5.*)

The use.

'This Reciprocation of the bases and heights 'makes the solid very easie to be measur'd. And 'the proposition seems to bear some analogy to 'the 14. *Prop. of the 6.* which asserts, That equiangular and equal parallelograms have their 'sides reciprocal; and the practice of the Rule 'of Three may be demonstrated from both.

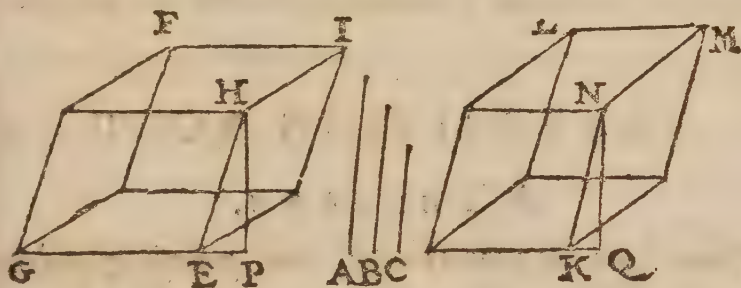
The 35. Prop. may be omitted.

PROP.

PROPOSITION XXXVI.

A THEOREM.

If three lines be in continual proportion, a parallelepipedon made of those three lines is equal to an equiangular parallelepipedon, which has all its sides equal to the middle line.



IF the lines A, B, C, be in continual proportion, the parallelepipedon EF made of those three lines, the side FI being equal to the line A, HE equal to B, and HI equal to C, is equal to the equiangular parallelepipedon KL, whose sides LM, MN, and KN, are each of them equal to the line B. From the points H and N draw the lines HP and NQ perpendicular to the planes of the bases; which lines will be equal, because the solid angles E and K are suppos'd equal, (so that if they could penetrate, neither would exceed the other,) and the lines EH

X x

and

and KN are also suppos'd equal. Therefore the heights HP and NQ are equal.

Demonstration.

There is the same proportion of A to B, as of FI to LM, as of B to C, or LM to HI: therefore the parallelogram FH contain'd under FI and IH is equal to the parallelogram LN contain'd under LM and MN both equal to B, (by the 16.6.) therefore the bases are equal. But the heights HP and NQ are also equal; therefore (by the 31.) the parallelepipeds are equal..

PROPOSITION XXXVII.

A THEOREM.

If four lines be proportional, the parallelepipeds describ'd upon those lines are proportional: and if the similar parallelepipeds be proportional, their homologous sides will be also proportional..

$$\frac{A}{C} = \frac{B}{D}$$

IF the line A has the same proportion to B as C to D, the similar parallelepipeds, whose homologous sides are the lines A, B, C, D, will be in the same proportion.

Demonstration.

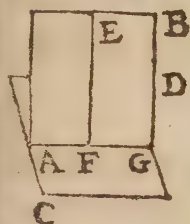
The parallelepipedon A is in the triplicate proportion to the parallelepipedon B, of the

of the line A to the line B, or that of the line C to the line D. But the parallelepipedon C to the parallelepipedon D is also in the triplicate proportion of that of the line C to the line D, (*by the 33.*) Therefore the parallelepipedon A has the same proportion to the parallelepipedon B, as the parallelepipedon C to the parallelepipedon D.

PROPOSITION XXXVIII.

A THEOREM.

If two planes be perpendicular to each other, a perpendicular drawn from a point in one of the planes to the other will fall upon the common section.



IF, the planes AB and CD being perpendicular to each other, you draw from the point E in the plane AB a line perpendicular to the plane CD, it will fall upon the common section of the planes.

Draw EF perpendicular to the common section AG.

Demonstration.

The line EF, perpendicular to AG, the common section of the planes, which are suppos'd to be perpendicular, will be perpendicular to the plane CD, (*by defin. 3.*) and because two lines

lines cannot be drawn from the point E perpendicular to the plane CD, (by the 13.) every perpendicular will fall upon the common section AG.

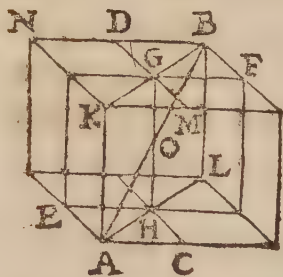
The USE.

'This proposition ought to have follow'd next after the 17th. because it respects solids in general. Tis of use to us in the Treatise of *Astrolabes*, to prove that in the *Analemma* all the circles, perpendicular to the *Meridian*, ought to be markt by right lines.

PROPOSITION XXXIX.

A THEOREM.

If in a parallelepipedon be drawn two planes, which divide the opposite sides into two equal parts, their common section and the diameter will also divide each other into two equal parts.



Suppose the opposite sides of the parallelepipedon AB to be divided into two equal parts by the planes CD and EF, their common section GH and the diameter BA will equally divide each other at the point O. Draw the lines BG, GK, AH and HL. I shall prove first, that the

TWO

two first of these, BG and GK, (and so likewise AH and HL,) make but one right line. For the triangles DGB and KMG have their sides DB and KM equal, because they are the halves of equal sides ; as also GD and GM. Further, DB and KM being parallel, the alternate angles BDG and GMK will be equal, (*by the 29.1.*) and therefore (*by the 4.1.*) the triangles DBG and KGM will be equal in all respects, and consequently the angles BGD and KGM ; and [*by the coroll. of the 15.1.*] BG and GK make but one right line, as also LH and HA : therefore ALBK is one plane, in which are found both the diameter AB, and the common section of the planes GH. The plane ALBK cutting the parallel planes AN and CD, their common sections GH and AK will be parallel : and [*by the 2.6.*] BG will have the same proportion to GK, as BO to OA ; and therefore [*by the 18.5.*] as BK to GK, so BA to BO ; and [*by the 4.6.*] so GH or AK to OG. But BK is double to BG, therefore BA is double to BO and AK, equal to GH, double to GO. Therefore the lines GH and AB divide each other equally at the point O.

Coroll. 1. All the diameters are divided at the point O.

Coroll. 2. Here we may adde some Corollaries, which depend upon divers propositions. As for example, that triangular prisms of the same

same hight are in the same proportion as their bases : for the parallelepipedons, of which they are the halves, are [*by the 32.*] in the same proportion as their bases : therefore the halves of their bases, and the halves of the parallelepipedons, *i. e.* the prisms, will be in the same proportion.

Coroll. 3. Polygon prisms of the same hight are also in the same proportion as their bases, because they may be resolv'd into triangular ones, each of which will have the same proportion as their bases.

Coroll. 4. The rest of the propositions concerning parallelepipedons are also applicable to prisms : as for example, that equal prisms have their hights and bases reciprocal ; and that similar prisms are in the triplicate proportion of that of their homologous sides.

The USE.

‘This proposition may help us to find out the
‘center of Gravity in parallelepipedons ; and to
‘demonstrate some other propositions in the
‘thirteenth and fourteenth books of *Euclid*,

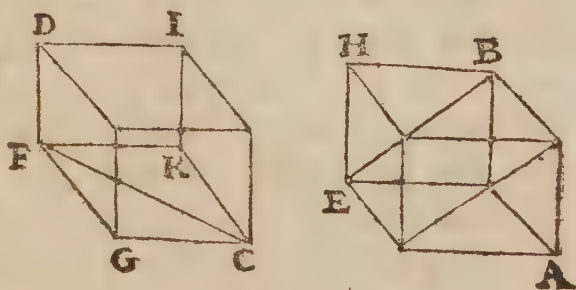
PROP.

PROPOSITION XL.

A THEOREM.

A Prism, that has a parallelogram for its base double to the triangular base of another prism, and of the same height, is equal to it.

LET ABE and CDG be two triangular prisms, of the same height; and the base of one the parallelogram AE, double to the triangle FGC, the base of the other prism: I say these prisms are equal. Suppose the parallelepipeds AH and GI were completed.



Demonstration.

'Tis suppos'd, that the base AE is double to the triangle FGC, but the parallelogram GK is double to the same triangle, [by the 34. 1.] therefore the parallelograms AE and GK are equal;

equal ; and consequently, the parallelepipeds AH and GI, having the same bases and the same heights, are equal ; and therefore the prisms, that are the halves, [*by the 26.*] will be likewise equal.



THE



THE TWELFTH BOOK
OF THE
ELEMENTS
OF
EUCLID.

E *uclid*, after having in the preceding books deliver'd the general principles of solid bodies, and explain'd the manner of measuring the most regular of them, that is, such as are terminated by plain superficies's; treats in this of such bodies as are contain'd in superficies's that are crooked, as the Cylinder, Cone, and Sphere: comparing one with the other, and giving rules, relating both to their solidity, and the manner of taking their dimensions. The book is of exceeding great use, because in it we find the principles upon which the most learned *Mathematicians* have built so many famous demonstrations concerning the Cylinder, the Cone, and the Sphere.

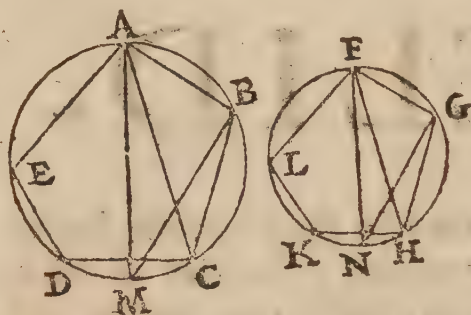
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PROP.

PROPOSITION I.

A THEOREM.

Similar polygons, inscrib'd in circles, are in the same proportion as the squares of the diameters of the same circles.



IF the polygons ABCDE, and FGHL, inscrib'd in circles, be similar, they will be in the same proportion

as the squares of the diameters AM, FN. Draw the lines BM, GN, AC, and FH.

Demonstration.

Tis suppos'd that the polygons are similar: that is to say, that the angles B and G are equal, and that AB has the same proportion to BC as FG to GH: from whence I infer, [by the 6.6.] that the triangles ABC and FGH are equiangular, and that the angles ACB and FHG are equal: so that likewise [by the 21.3.] the angles AMB and FNG are equal. But the angles ABM and FGN, being in a semicircle, are right angles, [by the 31.3.] and consequently the triangles ABM and FGN are equiangular.

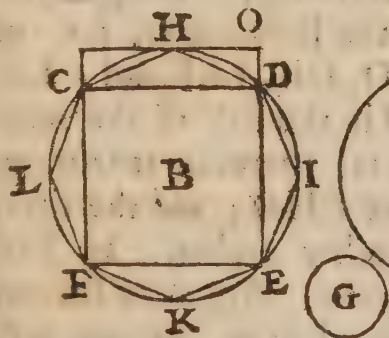
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Therefore [by the 4. 6.] AB has the same proportion to FG, as AM to FN : and [by the 22. 6.] if two similar polygons be describ'd upon AB and FG, as those that are propos'd ; and two other similar polygons upon AM and FN, which shall be two squares ; the polygon ABCDE will have the same proportion to the polygon FGHLK, as the square of AM to the square of FN.

"This proposition is necessary to demonstrate
"that which follows.

LEMMA.

If a certain quantity be less than a circle, a regular polygon may be inscrib'd in the same circle greater than that quantity.



Suppose the figure A to be less than the circle B ; a regular polygon may be inscrib'd in

the same circle, which shall be greater than the figure A. Let the figure G be the difference between the figure A and the circle, so that the figures

'figures A and G taken together, may be equal
 'to the circle B. Inscribe in the circle B the
 'square CDEF, [*by the 6. 4.*] and if the square
 'be greater than the figure A, we shall have
 'what we wanted. If it be less, divide the four
 'quarters of the circle CD, DE, EF, and FC,,
 'each into two equal parts at the points H, I, K,,
 'L, that so you may have an octagon. But if
 'the octagon be still less than the figure A,,
 'subdivide its archs, and you will have a polygon
 'of sixteen sides, afterwards of thirty two, and
 'then of sixty four. I say, at length you will have
 'a polygon greater than the figure A, *i. e.* a po-
 'lygon whose difference from the circle is less
 'than that of the figure A, that is, less than the
 'figure G. *Demonstration.*

'The inscrib'd square is more than half of the
 'circle, being half of the square describ'd about
 'the circle; and in describing the octagon you
 'take more than half of the Remainder, *i. e.* of
 'the four segments CHD, DIE, EKF, and CLF.
 'For the triangle CHD is the half of the rect-
 'angle CO, [*by the 34. 1.*] therefore it is more
 'than half of the segment CHD; and the same
 'may be said of all the other arches. In like
 'manner, in describing the polygon of sixteen
 'sides, you take more than half of what was left
 'of the circle; and so in all the others. There-
 'fore you will leave at last a less quantity than
 'G. For tis evident, that two unequal quanti-

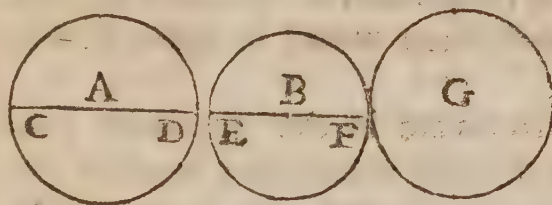
ties

'ties being propos'd, if you take away more
'than half of the greater, and afterwards more
'than half of what remains, and again more than
'half of what is still left behind ; at length that
'which remains will be less than the second
'quantity. Suppose the second quantity to be
'contain'd in the first an hundred times : tis
'evident, that dividing the first into an hundred
'parts, in such sort, that the first part may have
'a greater proportion to the second than two to
'one ; the last will be less than the hundredth
'part : so that at last you will obtain a polygon,
'which will be less exceeded by the circle, than
'the circle exceeds the figure A ; that is to say,
'that what will remain of the circle, when the
'polygon is taken away, will be less than G.
'Therefore the polygon will be greater than the
'figure A.

PROPOSITION II.

A THEOREM.

*Circles are in the same proportion as the squares
of their diameters.*



I Prove, that
the circles
A and B are
in the same
proportion, as
the

the squares of CD and EF. Suppose the figure G to have the same proportion to the circle B, as the square of CD to the square of EF: if the figure G be less than the circle A, [*by the preceding Lemma,*] a regular polygon may be inscrib'd in the circle A greater than G. Let a similar regular polygon be also inscrib'd in the circle B.

Demonstration.

The polygon of the circle A will have the same proportion to the polygon of B, as the square of CD to the square of EF, *i. e.* the same as G to the circle B; but the quantity G is less than the polygon inscrib'd in A: accordingly therefore [*by the 14. 5.*] the circle B must be less than the polygon inscrib'd in it, which is manifestly false. It must therefore be granted that the figure G, being less than the circle A, cannot have the same proportion to the circle B, as the square of CD to the square of EF; and consequently, that the circle A cannot have a greater proportion to the circle B, than the square of CD to the square of EF: Nor can it be said to have a less; for then the circle B would have a greater proportion to the circle A, and the same demonstration would be applicable to it.

Coroll. 1. Circles are in the duplicate proportion of that of their diameters; because the squares being similar figures, are in the duplicate

cate proportion of that of their sides, [by the 20.6.]

Coroll. 2. Circles are in the same proportion as the similar polygons, that are inscrib'd in them.

Coroll. 3. This ought to be well observ'd as a general rule : When similar figures, being inscrib'd in others, so that they may approach still nearer and nearer to them, and at last degenerate into the figures themselves, are in the same proportion ; the figures that contain them are also in the same proportion. What I would say is this ; That similar regular polygons, inscrib'd in divers circles, are always in the same proportion as the squares of the diameters ; and being made of more sides, so as to approach still nearer and nearer to the circles, they still retain the same proportion ; and the circles themselves are in the same proportion as the squares of their diameters. This manner of measuring round bodies, by inscribing in them others, is of great use.

The USE.

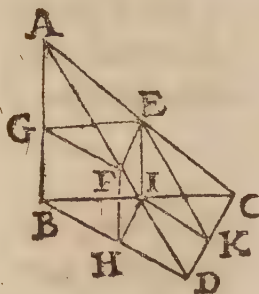
'This being a very general Proposition, enables us to argue about circles in the same manner as we do of squares. For example, we say [in the 47. 1.] That in a rectangle triangle the square of the base alone is equal to the squares of both the sides taken together. We may say the same of circles, *i. e.* That the circle, describ'd

'scrib'd upon the base of a rectangle triangle, is
'equal to the circles, whose diameters are the
'sides. And in the same manner may we aug-
'ment or diminish circles, according to what
'proportion we please. We prove also by it in
'Opticks, that the light decreases in the dupli-
'cate proportion of that of the distances of the
'lucid bodies.

PROPOSITION III.

A THEOREM.

Every Pyramid, whose base is triangular, may be divided into two equal prisms, which make up more than half of the pyramid; and into two equal pyramids.



IN the pyramid ABCD may be found two equal prisms, EBF I, and EHKC, which will be greater than half the pyramid. Divide the six sides of the pyramid equally at the points G, F, E, I, H, K, and draw the lines EG, GF, FE, EI, HI, FH, IK, and EK.

Demonstration.

In the triangle ABD, AG has the same proportion to GB as AF to FD, because AB and AD

AD are equally divided in G, and F ; therefore [by the 2.6.] GF and BD are parallels ; and GF will be the half of BD, *i. e.* equal to BH. In like manner, GE and BI, FE and HI, will be parallels, and equal : and [by the 15.11.] the planes GFE and BHI will be parallel, and consequently EBFI will be a prism. The same may be said of the figure HEKF, which will be also a prism equal to the other, [by the 40.11.] the parallelogram base HIKD being double the triangular BHI, [by the 41.1.]

Secondly, I say, the pyramids AEFG, and ECKI, are similar and equal.

Demonstration.

The triangles AFG and FDH are equal, [by the 8.11.] as also FDH and EIK ; and likewise AGE, and EIC, and so of all the other triangles of the pyramids : therefore the pyramids are equal, (by defin. 10.11.) They are also similar to the great pyramid ABDC : for the triangles AGE and ECI are similar, (by the 2.6.) the lines GE and BC being parallels ; and the like may be demonstrated of all the other triangles of the lesser pyramids.

Lastly, I say the prisms are more than half of the first pyramid. For if each was equal to one of the lesser pyramids, both would be equal to the half of the greater pyramid. But they are each of them greater than one of those pyramids ; as the prism GHE contains the pyramid

Z z

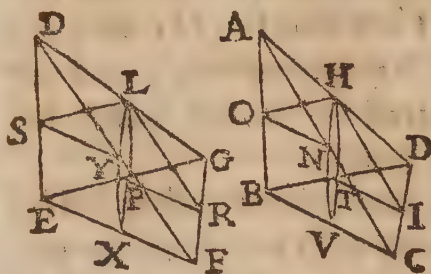
GBHI

GBH, and somewhat more; and that pyramid is equal and similar to the others, having all their triangles equal and similar to those of the pyramid AGFE, as may be easily prov'd by the parallelism of their sides: from whence I infer, that the two prisms taken together are greater than the two pyramids, and consequently greater than half of the great pyramid.

PROPOSITION IV.

A THEOREM.

If two triangular pyramids of the same height be divided into two prisms and two pyramids, and the latter pyramids subdivided after the same manner; all the prisms of one pyramid will have the same proportion to all those of the other, as the base of one pyramid to the base of the other.



IF the two pyramids ABCD, DEFG, of the same height, and having triangular bases, be divided into two prisms and two pyramids,

according to the method laid down in the third proposition; and the two lesser pyramids be subdivided after the same manner, and

so

so in order, that, having made as many divisions of one as of the other, you have the same number of prisms in both; I say, that all the prisms of one will have the same proportion to all the prisms of the other, as their bases.

Demonstration.

The pyramids being of the same height, the prisms, produc'd by the first division, will have also the same height, because they have each the half of that of their pyramids. But prisms of the same height are in the same proportion as their bases, (*by the coroll. of the 39. 11.*) The bases BTV and EPX are similar to the bases BDC and EGF; and having for their sides the half of those great bases, they can make but the fourth part of them, but they are in the same proportion as the great bases are; therefore the first prisms will have the same proportion as the great bases. After the same manner I may prove that the prisms produc'd by the second division, *i.e.* of the lesser pyramids, will be in the same proportion as the bases of those lesser pyramids, which are in the same proportion as the great bases. Therefore all the prisms of one have the same proportion to all the prisms of the other, as the base to the base.

The U S E.

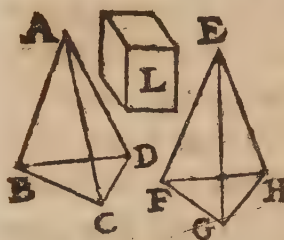
“These two propositions are necessary to
“compare pyramids together, and to take their
“dimensions.

PROP.

PROPOSITION V.

A THEOREM.

Triangular pyramids of the same height are in the same proportion as their bases.



THE pyramids ABCD and EFGH are in the same proportion as their bases. For if they were not, one of them, *e. g.* ABCD, would have a greater proportion to the pyramid EFGH, than the base BCD to the base FGH; so that a quantity less than ABCD would have the same proportion to the pyramid EFGH, as the base BCD to the base FGH. Divide the pyramid ABCD after the manner of the third proposition; divide also the pyramids, that result from that first division, into two prisms and two pyramids, and those again into two other prisms, continuing the division as long as there shall be occasion. Since the prisms of the first division are more than the half of the pyramid ABCD, (*by the 3.*) and the prisms of the second division more than half the remainder, *i. e.* of the two lesser pyramids, and those of the third division still more than the half of what is left; it is evident, that
so

so many divisions may be made, that that which remains shall be less than the excess of the pyramid $ABCD$ above the quantity L , that is, that all the prisms taken together shall be greater than the quantity L . Make as many divisions of the pyramid $EFGH$, so that you may have as many prisms as there are in $ABCD$. *Demonstration.*

The prisms of $ABCD$ have the same proportion to the prisms of $EFGH$, as the base BCD to the base FGH : but the proportion of the base BCD to the base FGH is the same with that of the quantity L to the pyramid $EFGH$: therefore the prisms of $ABCD$ have the same proportion to the prisms of $EFGH$, as the quantity L to the pyramid $EFGH$. But also the prisms of $ABCD$ are greater than the quantity L : therefore (*by the 14. 5.*) the prisms contain'd in the pyramid $EFGH$ would be greater than the same pyramid $EFGH$, which is evidently false, because the part cannot be greater than the whole. Therefore it must be granted, that no quantity less than one of the pyramids can have the same proportion to the other as the base to the base, and consequently neither of the pyramids can have a greater proportion to the other than the base to the base.

PROP

PROPOSITION VI.

A THEOREM

All sorts of pyramids, of the same height, have the same proportion as their bases.



THE pyramids ABC and DEFG, of the same height, are in the same proportion as the bases BC and EFG. Di-

vide the bases into triangles.

Demonstration.

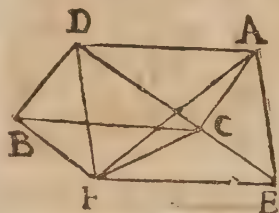
The triangular pyramids AB and DE, being of the same height, are in the same proportion as their bases, (*by the 5.*) So also the triangular pyramids AC and DF are in the same proportion as their bases. Therefore the pyramid ABC has the same proportion to the pyramid DEF, as the base BC to the base EF, (*by the 12.5.*) Further, since the pyramid DEF has the same proportion to the pyramid ABC, as the base EF to the base BC; and again, the pyramid DG has the same proportion to the pyramid ABC, as the base G to the base BC; the pyramid DEFG will also have the same proportion to the pyramid ABC, as the base EFG to the base BC.

PROP.

PROPOSITION VII.

A THEOREM.

Every pyramid is the third part of a prism, being upon the same base, and of the same height.



Suppose first the triangular prism AB be propos'd: I say, a pyramid, having one of the triangles ACE or BDF for its base, and being of the same height, as the pyramid ACEF, will be the third part of the prism. Draw the three diagonals AF, DC, FC, of the three parallelograms.

Demonstration.

The prism is divided into three equal pyramids, ACFE, ACFD, and CFBD; therefore each will be the third part of the prism. The two first, having for their bases the triangles AEF and AFD, which (*by the 34.1.*) are equal, and for their height the perpendicular drawn from the top C to the plane of their bases AF, will be equal, (*by the preceding.*) The pyramids ACFD, and CFBD, which for their bases have the equal triangles ADC and DCB, and the same top F, will be also equal, (*by the preceding.*) Therefore one of those pyramids, e.g. AFCE,

AFCE, having the same base BDF with the prism, and the same hight, which is the perpendicular drawn from the point *F* to the plane of the base ACE, is the third part of the same prism. If the prism be a polygon, it must be divided into divers triangular prisms; and the pyramid, which has the same base, and the same hight, will be also divided into as many triangular pyramids; each of which will be the third part of its prism. Therefore (*by the 12. 5.*) the polygon pyramid will be the third part of the polygon prism.

PROPOSITION VIII.

A THEOREM.

Similar pyramids are in the triplicate proportion of that of their homologous sides.

IF the pyramids be triangular, compleat the prisms, which will be also similar, because they will have certain planes the same with those of the pyramids. But the similar prisms are in the triplicate proportion of their homologous sides, [*by Coroll. 4. of the 39. 11.*] therefore the pyramids, which [*by the preceding*] are the third parts of the prisms, will be in the triplicate proportion of that their homologous sides.

If the pyramids be polygons, they must be reduc'd to triangular pyramids.

PROP.

PROPOSITION IX.

A THEOREM.

Equal pyramids have the hights and bases reciprocal, and those that have their hights and bases reciprocal, are equal.

IF two equal triangular pyramids be propos'd, make prisms upon the same base, and of the same hight. Since every prism is triple his pyramid, (*by the 7.*) they will also be equal. But equal prisms have their bases and hights reciprocal, (*by coroll. 4. of the 39. 11.*) therefore the bases and hights of the pyramids, which are the same with those of the prisms, will be also reciprocal

Secondly, if the bases and hights of the pyramids be reciprocal, the prisms will be equal, as also the pyramids, which are the third parts of the prisms.

If the pyramids propos'd be polygons, they must be reduc'd to triangular pyramids.

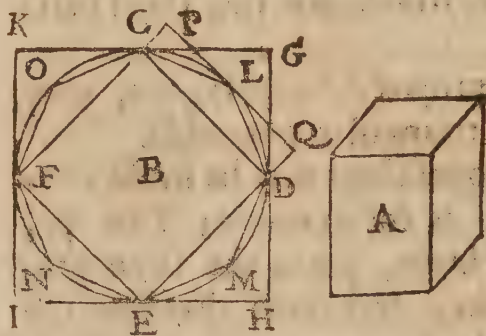
Coroll. Other propositions may be made concerning pyramids: as for example; That pyramids of the same hight, are in the same proportion as their bases; and those that have the same bases, are in the same proportion as their hights.

The USE.

'From these propositions is drawn the manner of measuring pyramids, which is, by multiplying their bases by the third part of their heights. Other propositions may also be made, as, That if a prism be equal to a pyramid, the bases and the height of the prism, with the third part of the height of the pyramid, will be reciprocal ; which is as much as to say, that if the base of the pyramid has the same proportion to the base of the prism, as the height of the prism to the third part of the height of the pyramid, the prism and the pyramid will be equal.

A L E M M A.

If a quantity less than a Cylinder be propos'd, a polygon prism may be inscrib'd in the Cylinder greater than that quantity.



IF the quantity A be less than the cylinder, whose base is the circle B, a polygon prism may be inscrib'd in the cylinder greater than the quantity A. The square

'square CDEF inscrib'd in, and GHIK circum-
'scrib'd about, the circle. CLDMENFO is an
'octagon inscrib'd. Draw the tangent PLQ: and
'suppose you had so many prisms as there are
'polygon bases, and all of the same height with
'the cylinder. That which has the circumscrib'd
'square for its base, will encompass the cylin-
'der; and that whose base is the inscrib'd square,
'will be also inscrib'd in the cylinder.

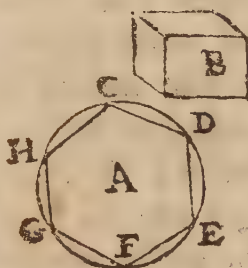
Demonstr. Prisms of the same height are in
'the same proportion as their bases, (*by coroll. 3.*
'*of the 39. 11.*) and the inscrib'd square being
'the half of that which is circumscrib'd, its
'prism will be the half of the other, and there-
'fore more than the half of the cylinder. And
'making the prism with the octagon base, you
'take away more than half of what remain'd of
'the cylinder, after the prism of the inscrib'd
'square was taken from it, because the trian-
'gle CLD is the half of the rectangle CQ. And
'because prisms of the same height are in the
'same proportion as their bases, the prism, whose
'base is the triangle CLD, will be the half of
'the prism, which for its base has the rectangle
'DCPQ: it will therefore be more than the
'half of that part of the cylinder, whose base is
'the segment DLC. The same may be said of
'all the other segments. After the same manner
'I may demonstrate, that making a polygon
'prism of sixteen sides, I take away more than
half

half of what remains of the cylinder, after the octagon prism is taken from it: so that there will remain at last a part of the cylinder, less than the excess of the cylinder above the quantity A. We shall have therefore a prism inscrib'd in the cylinder, which shall be less exceeded by the cylinder than the quantity A, *i.e.* which shall be greater than the quantity A. The same way of arguing will hold of the pyramids inscrib'd in a cone.

PROPOSITION X.

A THEOREM.

A Cone is the third part of a cylinder, having the same base, and being of the same height.



IF a cone and a cylinder have the circle A for their base, and be of the same height, the cylinder will be triple the cone. For if the proportion of the cylinder to the cone was greater than the triple proportion, the quantity B less than the cylinder would have the same proportion to the cone as three to one: and (*by the preceding Lemma*) a polygon prism may be inscrib'd in the cylinder greater than the quantity B. Suppose that which has for its base the polygon CDEFGH to be such an one. Make also upon the same base a pyramid inscrib'd in the cone. De.

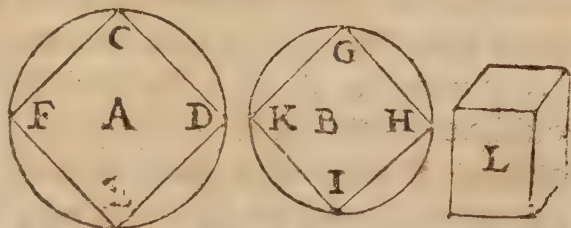
Demonstr. The cylinder, the cone, the prism, and the pyramid, are of the same height ; therefore the prism is the triple of the pyramid, (*by the 7.*) But the quantity B is also the triple of the cone ; therefore the prism has the same proportion to the pyramid, as the quantity B to the cone : and (*by the 14. 5.*) the prism being greater than the quantity B , the pyramid would be greater than the cone, in which it is inscrib'd, which is impossible.

But if it be said, that the cone has a greater proportion to the cylinder than one to three, the same method may be made use of to demonstrate the contrary.

PROPOSITION XI.

A THEOREM.

Cylinders and cones of the same height are in the same proportion as their bases.



LET two cones, or two cylinders, of the same height, be

propos'd, having for their bases the circles A and B ; I say, they are in the same proportion as their bases. For if not, one of them, *e. g.* the cylinder A would have a greater proportion to the

the cylinder B, than the base A has to the base B; suppose then that the quantity L, less than the cylinder A, has the same proportion to the cylinder B, as the base A to the base B. Therefore a polygon prism may be inscrib'd in the cylinder A, which shall be greater than the quantity L. Suppose it that therefore, whose base is the polygon CDEF; and inscribe a similar polygon GHIK in the base B, which is also the base of a cylinder of the same height.

Demonstr. The prisms of A and B are in the same proportion as their polygon bases, (*by coroll. 4. of the 39.11.*) and the polygons are in the same proportion as the circles, (*by coroll. 2. of the 2.*) therefore the prism A has the same proportion to the prism B, as the circle A to the circle B. But as the circle A to the circle B, so is the quantity L to the cylinder B: therefore as the prism A to the prism B, so is the quantity L to the cylinder B. But the prism A is greater than the quantity L, and consequently [*by the 14.5.*] the prism B, inscrib'd in the cylinder B, would be greater than its cylinder, which is impossible. Therefore neither of the cylinders has a greater proportion to the other, than its base to the other's base.

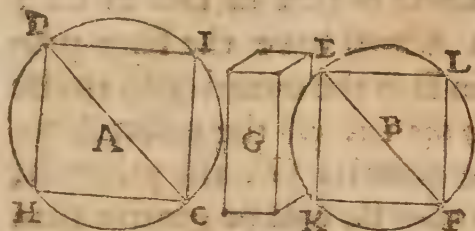
Coroll. Cylinders are triple the cones, of the same height, therefore cones of the same height are in the same proportion as their bases.

PROP.

PROPOSITION XII.

A THEOREM.

Cylinders and cones, that are similar, are in the triplicate proportion of that of the diameters of their bases.



LET two cones or two cylinders, that are similar, be propos'd, having the circles A and B for their bases; I say, that the proportion of the cylinder A to the cylinder B is the triplicate proportion of that of the diameter DC to the diameter EF. For if it be not the triplicate proportion; let the quantity G, less than the cylinder A, be, to the cylinder B, in the triplicate proportion of that of the diameter DC to the diameter EF; and inscribe a prism in the cylinder A greater than G, and another similar to it in the cylinder B: they will be of the same height with the cylinder, because similar cylinders have their heights and the diameters of their bases proportional, as well as prisms, (*by defin. 22.11.*)

Demonstr. The diameter DC has the same proportion to the diameter EF as the side DI to the side EL, or as DC to EF, as I have shown
in

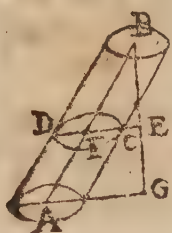
in the first. But similar prisms are in the triplicate proportion of that of their homologous sides, (*by coroll. 4. of 39. 11.*) therefore the prism A to the prism B is in the triplicate proportion of that of DC to EF. But we suppos'd that the quantity G in respect of the cylinder B was in the triplicate proportion of that of DC to EF; therefore the prism A will have the same proportion to the prism B as the quantity G to the cylinder B; and (*by the 14. 5.*) the prism A being greater than the quantity G, the prism B, inscrib'd in the cylinder B, will be greater than the cylinder B, which is impossible. Therefore similar cylinders are in the triplicate proportion of that of the diameters of their bases.

Cones are the third parts of Cylinders, [*by the 10.*] therefore similar cones are in the triplicate proportion of that of the diameters of their bases.

PROPOSITION XIII.

A THEOREM.

If a cylinder be cut by a plane, that is parallel to its base, the parts of its axis will be in the same proportion as the parts of the cylinder.



LET the cylinder AB be cut by the plane DC parallel to its base: I say, the cylinder AF will have the same proportion to the cylinder FB, as the line AF to the line

line FB . Draw the line BG perpendicular to the plane of the base A . Draw also upon the planes of the circles DC and A the lines FE and AG .

Demonstration.

The plane of the triangle BAG cuts the parallel planes A and DC ; therefore the sections FE and AG are parallel, (*by the 16.11.*) So that AF has the same proportion to FB , as the height GE to EB . Take any aliquot part of EB ; and having divided GE and EB into parts equal to it, draw so many planes parallel to the base A ; then will you have so many cylinders of the same height; which, having their bases and heights equal, will be equal, (*by the 11.*)

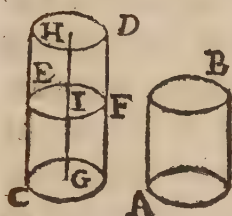
Further, the lines AF and FB will be divided after the same manner as EG and EB , [*by the 17.11.*] so that the line AF will as oft contain any aliquot part of the line FB , as the cylinder AF contains the like aliquot part of the cylinder FB ; therefore the parts of the cylinder will be in the same proportion as the parts of their axis.

Coroll. The parts of the perpendicular are in the same proportion as the parts of the cylinder.

PROPOSITION XIV.

A THEOREM.

Cylinders and cones, having the same bases, are in the same proportion as their heights.



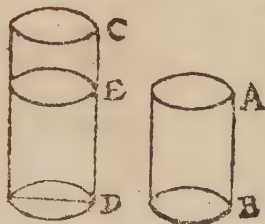
TWO cylinders of equal bases being propos'd, as AB and CD , cut in the greater a cylinder of the same height with the less, drawing a plane EF parallel to its base. 'Tis evident [*by the 11.*] that the cylinders CF and AB are equal; and that CF has the same proportion to CD , as GI to GH , or [*by the coroll. of the preceding*] as the height of CF to the height of CD ; therefore AB has the same proportion to CD , as the height of CF or AB to the height of GD .

Cones, being the third parts of cylinders, if their bases be equal, will be also in the same proportion as their heights.

PROPOSITION XV.

A THEOREM.

Cylinders and cones that are equal, have their bases and heights reciprocal: and those, that have their bases and heights reciprocal, are equal.



IF the cylinders AB and CD be equal, the base B will have the same proportion to the base D , as the height CD

to

to the hight AB. Take the hight DE equal to the hight AB.

Demonstr. The cylinder AB has the same proportion to the cylinder DE, of the same hight, as the base B to the base D, [by the 11.] But as the cylinder AB is to the Cylinder DE, so is the cylinder CD, equal to AB, to the cylinder DE; *i.e.* so is the hight CD to the hight AB or DE. Therefore as the base B to the base D, so is the hight CD to the hight AB.

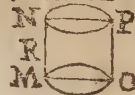
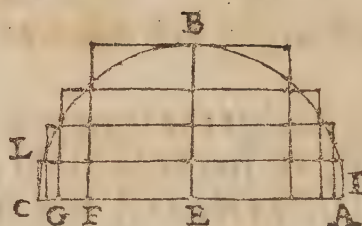
Secondly, if the base B has the same proportion to the base D, as the hight CD to the hight AB, the cylinders AB and CD will be equal. For the cylinder AB is in the same proportion to the the cylinder DE, as the base B to the base D: and the cylinder CD will have the same proportion to DE, as the hight CD to the hight DE: therefore AB has the same proportion to DE, as CD to DE; and [by the 9.5.] the cylinders AB and CD will be equal.

“The 16. and 17. Propositions are very difficult, and of no other use but to prove the 18. which may more easily be done by the following Lemma’s,

LEMMA 1.

If a quantity be propos’d less than a sphere, cylinders of the same hight may be inscrib’d in the same sphere greater than that quantity.

sup-



Suppose A
 BC to be
 a great semi-
 circle of the
 sphere; where-
 of we treat,
 and the quantity D to be the quantity less
 than that sphere: I say, several cylinders of
 the same height may be inscrib'd in the sphere,
 which taken together will be greater than the
 quantity D. For if the semi-sphere exceed the
 quantity D, it will exceed it by some magni-
 tude; let it then be the cylinder MP, so that
 the quantities D and MP taken together may
 be equal to the semi-sphere. Make a great cir-
 cle of the sphere to have the same proportion
 to the base MO, as the height MN to the height
 R. Then divide the line EB into as many e-
 qual parts as you please, each being less than
 R: and drawing parallels to the line AG, de-
 scribe the inscrib'd and circumscrib'd paralle-
 lograms. The number of the circumscrib'd
 will exceed that of the inscrib'd by one. But all
 the rectangles circumscrib'd will surpass all the
 inscrib'd by the little rectangles through
 which the circumference of the circle passes;
 all which taken together are equal to the rect-
 angle AL. I imagine then the semicircle to
 be turn'd about upon the diameter EB; the se-
 micircle will by that motion describe a semi-
 sphere,

'sphere, and the rectangles inscrib'd so many
'cylinders inscrib'd in the semisphere; and the
'circumscrib'd, other cylinders circumscrib'd.

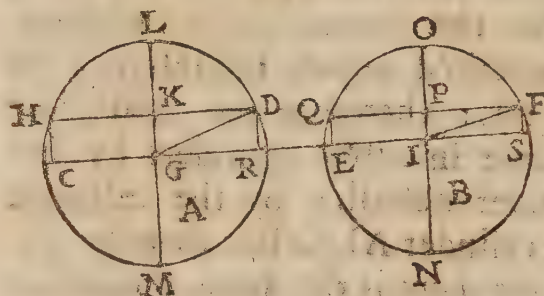
Demonstr. The circumscrib'd cylinders sur-
'pass the inscrib'd more than the semi-sphere
'surpasses the same inscrib'd cylinders, it being
'contain'd within the circumscrib'd cylinders.
'But the circumscrib'd surpass the inscrib'd by
'the cylinder AL: therefore the semisphere
'will surpass those inscrib'd cylinders by less
'than the cylinder describ'd by the rectangle
'AL. But the cylinder AL is less than the cy-
'linder MP: for there is the same proportion
'of a great circle of the sphere, which is the
'base of the cylinder AL, to MO, as of MN to
'R; therefore (*by the preceding*) a cylinder,
'which should have a great circle of the sphere
'for its base, and the hight R, would be equal
to the cylinder MP: but the cylinder AL, tho
'it have the same base, yet its hight CL is less
'than R; therefore the cylinder AL is less than
'the cylinder MP. Consequently the semi-
'sphere, that exceeds the quantity D by the cy-
'linder MP, and the inscrib'd cylinders by a
'quantity less than AL; exceeds the inscrib'd
'cylinders by less, than it exceeds the quantity
'D; therefore the quantity D is less than the
'cylinders inscrib'd in the semisphere.

'That which I have said of the semi-sphere, is
'applicable to an entire sphere.

L E M-

LEMMA II.

Similar cylinders, inscrib'd in two spheres, are in the triplicate proportion of the diameters of the spheres.



IF two similar cylinders CD and EF be inscrib'd in the spheres A and B,

they will be in the triplicate proportion of the diameters LM and NO. Draw the lines GD and IF.

Demonstration.

The right cylinders CD and EF are similar; therefore HD has the same proportion to DR as QF to FS, as also KD has the same proportion to DG as PF to FI. Consequently the triangles GDK and IFP are similar, (*by the 6, 6.*) therefore KD has the same proportion to PF as GD to IF, or LM to ON. But the similar cylinders CD and EF are in the triplicate proportion of KD and PF, the semidiameters of their bases, (*by the 12.*) therefore the similar cylinders CD and EF, inscrib'd in the spheres A and B, are in the triplicate proportion of the diameters of the spheres.

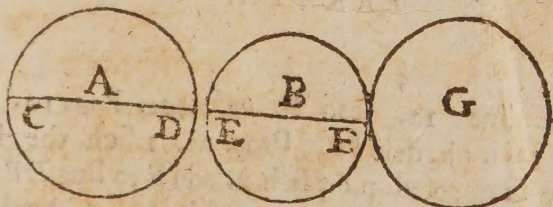
PROP.

PROPOSITION XVIII.

A THEOREM.

Spheres are in the triplicate proportion of their diameters.

THE spheres A and B are in the triplicate proportion of that of their diameters CD and EF. For if not, one of the spheres, suppose A, will be in a greater proportion to B, than the triplicate of that of the diameters CD and EF; therefore the quantity G, less than the sphere A, will be in the triplicate proportion of that of CD to EF, to the sphere B; and then some cylinders may (*according to the Lem. 1*) be inscrib'd in the sphere A, greater than the quantity G. Inscribe an equal number of cylinders in the sphere B, similar to those in the sphere A.



Demonstr. The cylinders of the sphere A to those of the sphere B are in the triplicate proportion of that of CD to EF, but the quantity G to the sphere B is also in the triplicate proportion of that of CD to EF: therefore the cylinders of the sphere A have the same proportion to the similar cylinders of the sphere B, as the

the quantity G to the sphere B . Consequently the cylinders of A being greater than the quantity G , the cylinders of B , i. e. inscrib'd in the sphere B , will be greater than the sphere B , which is impossible. Therefore the spheres A and B are in the triplicate proportion of that of their diameters.

Coroll. Spheres are in the same proportion as the cubes of their diameters; because cubes, being similar solids, are in the triplicate proportion of their sides, [by the 33.11.]

The END.

ERRATA.

Pag. 1. line 12. read *Propositions*. Pag. 10. l. 8. add *An.* p. 24. l. ult. dele *be*. Prop. 22. 1 Sch. the letters A and B are Transposed. p. 64 Sch. 2. add F to line EB produced below CD . p. 78, l. 15. r. *not*. l. 23. r. by the 37. pag. 99. l. penult. dele *of*. p. 101. l. 2. for EC read FB . pag. 151. l. 6. for AE r. AI . p. 167. l. 3. for EF r. DF . p. 180. l. 10. for EH r. FH . p. 199. l. 10. del. *of* p. 209. l. 9. for *had* read *have* p. 260. l. 13. for AC r. EC . l. 17. for AB r. DB . p. 263. Sch. transfer A to the other side of the scheme.

abed
efgh



